

Reflected BSDEs on Filtered Probability Spaces

Tomasz Klimsiak

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University
Chopina 12/18, 87–100 Toruń, Poland
e-mail: tomas@mat.uni.torun.pl

Abstract

We study the problem of existence and uniqueness of solutions of backward stochastic differential equations with two reflecting irregular barriers, L^p data and generators satisfying weak integrability conditions. We deal with equations on general filtered probability spaces. In case the generator does not depend on the z variable, we first consider the case $p = 1$ and we only assume that the underlying filtration satisfies the usual conditions of right-continuity and completeness. Additional integrability properties of solutions are established if $p \in (1, 2]$ and the filtration is quasi-continuous. In case the generator depends on z , we assume that $p = 2$, the filtration satisfies the usual conditions and additionally that it is separable. Our results apply for instance to Markov-type reflected backward equations driven by general Hunt processes.

1 Introduction

In the present paper we study the problem of existence and uniqueness of solutions of backward stochastic differential equations (BSDEs for short) with two reflecting barriers. There is now an extensive literature on the subject, but unfortunately all the available results concern equations with underlying filtration generated by a Wiener process (Brownian filtration) or by a Poisson random measure and an independent Wiener process (Brownian-Poisson filtration). In the paper we deal with equations on general filtered probability spaces. In the case where the generator of the equation is independent of the z variable we only assume that the underlying filtration $\mathcal{F} = \{\mathcal{F}_t; t \in [0, T]\}$ satisfies the usual conditions of right-continuity and completeness. In the general case of equations with generators depending on z we assume that the Hilbert space $L^2(\mathcal{F}_T)$ is separable. Another dominant feature of the paper is that we study equations with irregular barriers, L^p data ($p \in [1, 2]$ in case the generator is independent of z and $p = 2$ in the general case) and with generators satisfying weak regularity and growth assumptions.

In the case of Brownian filtration the theory of reflected BSDEs with irregular barriers and weak assumptions on the data is quite well developed. We refer the reader to [5, 16, 21] for existence and uniqueness results for equations with irregular barriers. Reflected BSDEs with monotone generator satisfying weak growth condition are studied

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in [11, 10, 15, 24], whereas equations with L^p -data and $p \in [1, 2]$ in [7, 11, 10, 24]. In the case of the Brownian-Poisson filtration the only known results concern reflected BSDEs with càdlàg barriers, Lipschitz-continuous generators and L^2 data (see [6, 8]).

Let $(\Omega, \mathcal{F} = \{\mathcal{F}_t; t \in [0, T]\}, P)$ be a filtered probability space satisfying the usual conditions. Suppose we are given an \mathcal{F}_T measurable random variable ξ (terminal time), a measurable function $f : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ (generator) such that $f(\cdot, y) \in \text{Prog}([0, T] \times \Omega)$ and two progressively measurable processes L, U (barriers) such that $L_t \leq U_t$ for a.e. $t \in [0, T]$. By a solution of the reflected BSDE with data ξ, f and barriers U, L (RBSDE(ξ, f, L, U)) for short) on (Ω, \mathcal{F}, P) we mean a triple (Y, M, R) consisting of an adapted càdlàg process Y of Doob's class (D), a local martingale M with $M_0 = 0$ and a predictable finite variation process R with $R_0 = 0$ such that

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dR_r - \int_t^T dM_r, \quad t \in [0, T], \quad (1.1)$$

$$L_t \leq Y_t \leq U_t \quad \text{for a.e. } t \in [0, T] \quad (1.2)$$

and the following minimality condition for R is satisfied: for every càdlàg processes \hat{L}, \hat{U} such that $L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$ we have

$$\int_0^T (Y_{t-} - \hat{L}_{t-}) dR_t^+ = \int_0^T (\hat{U}_{t-} - Y_{t-}) dR_t^- = 0, \quad (1.3)$$

where $R = R^+ - R^-$ is the Jordan decomposition of the measure dR . Condition (1.3) has been considered in [21] in the case of reflected BSDEs with Brownian filtration. Note that the above definition of a solution is similar in spirit to the definition of a solution of nonreflected BSDEs on general filtered spaces considered in [13, 17]. It is well suited for studying by probabilistic methods partial differential equations with irregular data (see [13, 14]).

In the paper we assume that f is continuous, monotone with respect to y and satisfies the following mild growth condition

$$E \int_0^T |f(t, 0)| dt < \infty, \quad \forall y \in \mathbb{R}, \quad [0, T] \ni t \mapsto f(t, y) \in L^1(0, T). \quad (1.4)$$

Condition (1.4) has appeared before in the papers on nonreflected (see [1]) and reflected (see [11, 10]) BSDEs with L^1 data adapted to the Brownian filtration. As for the barriers, we merely assume that they are measurable and satisfy the following Mokobodski condition: there exists a special semimartingale X with integrable finite variation part such that

$$L_t \leq X_t \leq U_t \quad \text{for a.e. } t \in [0, T], \quad E \int_0^T |f(t, X_t)| dt < \infty. \quad (1.5)$$

We prove that if f, L, U satisfy conditions (1.4), (1.5) and the data are in L^1 , i.e. $\xi \in L^1(\mathcal{F}_T)$ and $\int_0^T |f(\cdot, 0)| dt \in L^1(\mathcal{F}_T)$, then there exists a unique solution (Y, M, R) of RBSDE(ξ, f, L, U). We also show that under the assumptions ensuring the existence of a solution of nonreflected BSDE condition (1.5) is necessary for the existence of a solution of (1.1) such that $E|R|_T < \infty$. Furthermore, we show that if the jumps of the barriers are totally inaccessible and \mathcal{F} is quasi-left continuous then R is continuous

(reflected BSDEs with such barriers and Poisson-Brownian filtration are considered in [6]). Finally, we show that if the barriers satisfy the standard Mokobodski condition, i.e. only the first condition in (1.5) is satisfied, then the solution still exists but in general R is not integrable (it may happen that $E|R|_T^q = \infty$ for every $q > 0$, see [10]). In Section 5 we show that under the additional assumption of quasi-left continuity of the filtration \mathcal{F} , if the data are L^p integrable for some $p \in (1, 2]$, i.e. $\xi \in L^p(\mathcal{F}_T)$, $\int_0^T |f(\cdot, 0)| dt \in L^p(\mathcal{F}_T)$, $X \in \mathcal{H}^p$ and $\int_0^T |f(t, X_t)| dt \in L^p(\mathcal{F}_T)$, then the solution (Y, M) of (1.1)–(1.3) belongs to the space $\mathcal{S}^p \otimes \mathcal{M}^p$.

In the last section of the paper we study BSDEs with generators possibly depending on the z variable. To deal with such equations we need some sort of the representation theorem for square integrable martingales. In the paper we use the representation by series of stochastic integrals, because it applies to general filtered spaces. In the context of BSDEs this type of representation of martingales has been used in [2, 19]. It should be stressed, however, that our methods also works for other type of representations. For instance, one can employ the representation of [26], which is commonly used in the case of BSDEs with the Brownian-Poisson filtration (see Remark 6.5).

To have the representation theorem, we assume additionally that $L^2(\Omega, \mathcal{F}_T, P)$ is a separable Hilbert space. It is known that then there exists an orthogonal sequence $\{M^i\}$ of square integrable martingales such that each locally square integrable martingale N admits the representation

$$N_t = N_0 + \sum_{i=1}^{\infty} \int_0^t Z_r^i dM_r^i, \quad t \in [0, T] \quad (1.6)$$

for some sequence $\{Z^i\}$ of predictable processes. By a solution of $\text{RBSDE}(\xi, f, L, U)$ we mean a triple (Y, Z, R) consisting of a càdlàg adapted process $Y \in \mathcal{S}^2$, predictable process $Z = \{Z^i\}$ satisfying $P(\sum_{i=1}^{\infty} \int_0^T |Z_t^i|^2 d\langle M^i \rangle_t < \infty) = 1$ and a predictable finite variation process R with $R_0 = 0$, such that (1.2), (1.3) hold true and

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T dR_r - \sum_{i=1}^{\infty} \int_t^T Z_r^i dM_r^i, \quad t \in [0, T].$$

In our main result of the last section we assume that the data are in L^2 , i.e. $\xi \in L^2(\mathcal{F}_T)$ and $\int_0^T |f(\cdot, 0, 0)| dt \in L^2(\mathcal{F}_T)$, and that the generator f is monotone with respect to y and Lipschitz continuous with respect to z (in the appropriate norm generated by the sequence $\{M^i\}$), satisfies the growth condition with respect to y similar to (1.4) and

$$L_t \leq X_t \leq U_t \quad \text{for a.e. } t \in [0, T], \quad E\left(\int_0^T |f(t, X_t, 0)| dt\right)^2 < \infty$$

for some semimartingale $X \in \mathcal{H}^2$. We show that under these assumptions there exists a unique solution $(Y, Z, R) \in \mathcal{S}^2 \otimes \mathcal{M}^2 \otimes \mathcal{V}^2$ of $\text{RBSDE}(\xi, f, L, U)$.

We close the presentation of our main results with the following general remarks. In the existing literature mainly reflected BSDEs on spaces equipped with Brownian or Brownian-Poisson filtration are considered. One of the reasons is that such a framework is sufficient for applications of BSDEs to mathematical finance. In the present paper we improve these results on RBSDEs by relaxing assumptions on the generator. Namely,

we replace the Lipschitz continuity of f with respect to y by monotonicity and as a growth condition we only impose very weak integrability condition (1.4). Moreover, if the generator is independent of z , we consider L^p data for $p \in [1, 2)$. Reflected BSDEs on more general filtered probability spaces arise naturally in applications to partial differential equations involving operators generated by semi-Dirichlet forms or generalized Dirichlet forms. The papers [13, 14] show that BSDEs provide very efficient tool for investigating abstract elliptic equations of the form

$$-Au = f(x, u) + \mu, \quad (1.7)$$

where μ is a smooth measure and A is a Dirichlet operator. Similar to (1.7) parabolic equations are investigated in [12]. To study (1.7) one needs to consider backward equations with forward driving process being a general special standard Markov processes. Our main motivation for studying BSDEs in an abstract framework was to cover this class of processes. Also note that in the whole paper we consider generalized reflected RBSDEs, i.e. equations of the form (1.1) perturbed by some finite variation process V . In applications to PDEs we have in mind, adding V to (1.1) allows one to study equations with true measure data (V is then the additive functional of the forward process in the Revuz correspondence with the measure on the right-hand side of the equation).

2 BSDEs with one reflecting barrier

Let us fix $T > 0$ and a stochastic basis $(\Omega, \mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}, P)$ satisfying the usual conditions of right continuity and completeness. By \mathcal{T} we denote the set of all \mathcal{F} stopping times with values in $[0, T]$. For $s, t \in [0, T]$ such that $s \leq t$ we denote by \mathcal{T}_t (resp. $\mathcal{T}_{s,t}$) the set of all $\tau \in \mathcal{T}$ such that $P(\tau \in [t, T]) = 1$ (resp. $P(\tau \in [s, t]) = 1$).

By \mathcal{M} (resp. \mathcal{M}_{loc}) we denote the space of \mathcal{F} martingales (resp. local \mathcal{F} martingales). \mathcal{M}_0 (resp. \mathcal{M}^p) is the subspace of $M \in \mathcal{M}$ consisting of M such that $M_0 = 0$ (resp. $E[M]_T^{p/2} < \infty$). By \mathcal{V} (resp. \mathcal{V}^+) we denote the space of \mathcal{F} progressively measurable processes of finite variation (resp. increasing). \mathcal{V}_0 (resp. \mathcal{V}^p) is the subspace of \mathcal{V} consisting of V such that $V_0 = 0$ (resp. $E|V|_T^p < \infty$). ${}^p\mathcal{V}$ is the space of predictable processes in \mathcal{V} . By \mathcal{S}^p we denote the space of \mathcal{F} progressively measurable processes Y such that $E \sup_{t \leq T} |Y_t|^p < \infty$. By $L^p(\mathcal{F})$ we denote the space of \mathcal{F} progressively measurable processes X such that $E \int_0^T |X_t|^p dt < \infty$. $L^p(\mathcal{F}_T)$ is the space of \mathcal{F}_T measurable random variables X such that $E|X|_T^p < \infty$.

In the rest of the paper ξ is an \mathcal{F}_T measurable random variable, L is an \mathcal{F} -progressively measurable process, $V \in \mathcal{V}_0$, $f : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that $f(\cdot, y)$ is \mathcal{F} progressively measurable for every $y \in \mathbb{R}$. We also adopt the convention that every càdlàg process Y on $[0, T]$ extends to $[0, \infty)$ by $Y_t = Y_T$, $t \geq T$.

Definition. We say that a triple of processes (Y, M, K) is a solution of reflected backward stochastic differential equation with terminal condition ξ , right-hand side $f + dV$ and lower barrier L (RBSDE($\xi, f + dV, L$) for short) if

- (a) Y is \mathcal{F} adapted càdlàg process of Doob's class (D), $M \in \mathcal{M}_{0,loc}$, $K \in {}^p\mathcal{V}_0^+$,
- (b) $Y_t \geq L_t$ for a.e. $t \in [0, T]$,

(c) for every càdlàg process \hat{L} such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T]$,

$$\int_0^T (Y_{t-} - \hat{L}_{t-}) dK_t = 0,$$

(d) $[0, T] \ni t \rightarrow f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T dK_r - \int_t^T dM_r, \quad t \in [0, T].$$

We shall prove a comparison result (and consequently uniqueness) for solutions of reflected equations under the following monotonicity condition.

(H1) There is $\mu \in \mathbb{R}$ such that for a.e. $t \in [0, T]$ and every $y, y' \in \mathbb{R}$,

$$(f(t, y) - f(t, y'))(y - y') \leq \mu |y - y'|^2.$$

Proposition 2.1. *Let (Y^i, M^i, K^i) be a solution of RBSDE $(\xi^i, f^i + dV^i, L^i)$, $i = 1, 2$. Assume that $\xi^1 \leq \xi^2$, $L_t^1 \leq L_t^2$ for a.e. $t \in [0, T]$, $dV^1 \leq dV^2$, and either f^1 satisfies (H1) and $f^1(t, Y_t^2) \leq f^2(t, Y_t^2)$ for a.e. $t \in [0, T]$ or f^2 satisfies (H1) and $f^1(t, Y_t^1) \leq f^2(t, Y_t^1)$ for a.e. $t \in [0, T]$. Then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$.*

Proof. Set $Y = Y^1 - Y^2$, $M = M^1 - M^2$, $K = K^1 - K^2$. By the assumptions and the Tanaka-Meyer formula, for every $\tau \in \mathcal{T}$ we have

$$\begin{aligned} Y_\tau^+ &\leq Y_\tau^+ + \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} (f^1(r, Y_r^1) - f^2(r, Y_r^2)) dr + \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} d(V^1 - V^2)_r \\ &\quad + \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dK_r - \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dM_r \\ &\leq Y_\tau^+ + \mu \int_t^\tau Y_r^+ dr + \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dK_r^1 - \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dM_r. \end{aligned} \quad (2.1)$$

Observe that $L_t^1 \leq Y_t^1 \wedge Y_t^2 \leq Y_t^1$ for a.e. $t \in [0, T]$ and

$$\int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dK_r^1 = \int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} |Y_{r-}^1 - Y_{r-}^2|^{-1} (Y_{r-}^1 - Y_{r-}^1 \wedge Y_{r-}^2) dK_r^1.$$

Hence

$$\int_t^\tau \mathbf{1}_{\{Y_{r-}^1 > Y_{r-}^2\}} dK_r^1 \leq 0, \quad t \in [0, \tau]$$

by condition (c) of the definition of the solution of RBSDE $(\xi^1, f^1 + dV^1, L^1)$. From (2.1) it therefore follows that

$$Y_t^+ \leq Y_\tau^+ + \mu \int_t^\tau Y_r^+ dr - \int_t^\tau \mathbf{1}_{\{Y_{r-} > 0\}} dM_r, \quad t \in [0, \tau].$$

Let $\{\tau_n\} \subset \mathcal{T}$ be a fundamental sequence for the martingale M . Then by the above inequality,

$$EY_{t \wedge \tau_n}^+ \leq EY_{\tau_n}^+ + \mu E \int_{t \wedge \tau_n}^{\tau_n} Y_r^+ dr, \quad t \in [0, T].$$

Since Y is of class (D), letting $n \rightarrow \infty$ gives

$$EY_t^+ \leq \mu \int_t^T EY_r^+ dr, \quad t \in [0, T],$$

so applying Gronwall's lemma yields the desired result. \square

Corollary 2.2. *Let assumption (H1) hold. Then there exists at most one solution of RBSDE($\xi, f + dV, L$).*

Let us recall that a filtration \mathcal{F} is called quasi-left continuous if for every sequence $\{\tau_n\} \subset \mathcal{T}$ and $\tau \in \mathcal{T}$, if $\tau_n \nearrow \tau$ then $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\tau_n} = \mathcal{F}_\tau$.

Proposition 2.3. *Assume that \mathcal{F} is quasi-left continuous, $\xi \in L^1(\mathcal{F}_T)$, $V \in \mathcal{V}_0^1$ and L is a càdlàg process of class (D). Set*

$$Y_t = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} E \left(\int_t^\tau dV_r + L_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t \right). \quad (2.2)$$

Then for every predictable $\tau \in \mathcal{T}$ such that $\tau > 0$,

$$Y_{\tau-} = L_{\tau-} \vee (Y_\tau + \Delta V_\tau). \quad (2.3)$$

Proof. For simplicity we assume that τ is constant. The proof in the general case goes through as in case $\tau \equiv \text{const}$, with some obvious changes.

First observe that the process $\bar{Y}_t \equiv Y_t + \int_0^t dV_r$, $t \in [0, T]$, is a supermartingale and if we put

$$\bar{L}_t = L_t + \int_0^t dV_r, \quad \bar{\xi} = \xi + \int_0^T dV_r$$

then

$$\bar{Y}_{t-} = \bar{L}_{t-} \vee \bar{Y}_t \quad \text{iff} \quad Y_{t-} = L_{t-} \vee (Y_t + \Delta V_t).$$

Therefore without loss of generality we may and will assume that $V = 0$. Then Y is a supermartingale. By the assumptions of the proposition, Y is of class (D). Therefore there exist $K \in \mathcal{V}_0^{1,+}$ and $M \in \mathcal{M}_0$ such that

$$Y_t = Y_0 - K_t + M_t, \quad t \in [0, T].$$

Since \mathcal{F} is quasi-left continuous, $\Delta M_t = 0$, P -a.s. for every $t \in [0, T]$. Hence $\Delta Y_t \leq 0$ for every $t \in (0, T]$, which implies that for every $t \in (0, T]$,

$$Y_t \leq Y_{t-}. \quad (2.4)$$

On the other hand, $Y_t \geq L_t$, $t \in [0, T]$. Therefore $L_{t-} \leq Y_{t-}$, $t \in (0, T]$. From this and (2.4) it follows that for every $t \in (0, T]$,

$$Y_t \vee L_{t-} \leq Y_{t-}.$$

We are going to show the opposite inequality. To this end, let us fix $t \in (0, T]$ and $s \in [0, t)$. By known properties of Snell's envelope (see [4]),

$$\begin{aligned} Y_s &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{s,t}} E(L_\tau \mathbf{1}_{\{\tau < t\}} + Y_t \mathbf{1}_{\{\tau = t\}} | \mathcal{F}_s) \\ &= E(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{s,t}} (L_\tau \mathbf{1}_{\{\tau < t\}} + Y_t \mathbf{1}_{\{\tau = t\}}) | \mathcal{F}_s) \\ &\leq E(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{s,t}} (L_\tau \mathbf{1}_{\{\tau < t\}} + (Y_t \vee L_{t-}) \mathbf{1}_{\{\tau = t\}}) | \mathcal{F}_s) \equiv U_s. \end{aligned}$$

Observe that

$$Y_s \leq U_s, \quad s \in (0, t) \quad (2.5)$$

and

$$U_s = E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} (L_\tau \mathbf{1}_{\{\tau < t\}} + (Y_t \vee L_{t-}) \mathbf{1}_{\{\tau = t\}}) | \mathcal{F}_s) \geq E(Y_t \vee L_{t-} | \mathcal{F}_s),$$

which implies that

$$E(Y_t \vee L_{t-} | \mathcal{F}_{t-}) \leq \liminf_{s \rightarrow t^-} U_s.$$

Put

$$\hat{L}_r = L_r \mathbf{1}_{\{r < t\}} + (Y_r \vee L_{r-}) \mathbf{1}_{\{r = t\}}, \quad r \in [0, t].$$

With this notation,

$$U_s = E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} \hat{L}_\tau | \mathcal{F}_s).$$

For $\varepsilon > 0$ set

$$B_\varepsilon^s = \{ \sup_{s \leq r \leq t} \hat{L}_r \leq \hat{L}_t + \varepsilon \}.$$

Since \hat{L} is càdlàg and has nonnegative jump at $r = t$,

$$\lim_{s \rightarrow t^-} B_\varepsilon^s = \Omega. \quad (2.6)$$

We have

$$\begin{aligned} E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} \hat{L}_\tau | \mathcal{F}_s) &= E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} \hat{L}_\tau \mathbf{1}_{B_\varepsilon^s} | \mathcal{F}_s) + E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} \hat{L}_\tau \mathbf{1}_{(B_\varepsilon^s)^c} | \mathcal{F}_s) \\ &\leq E((\hat{L}_t + \varepsilon) \mathbf{1}_{B_\varepsilon^s} | \mathcal{F}_s) + E(\text{ess sup}_{\tau \in \mathcal{T}_{s,t}} \hat{L}_\tau \mathbf{1}_{(B_\varepsilon^s)^c} | \mathcal{F}_s). \end{aligned} \quad (2.7)$$

Since \hat{L} is of class (D), it follows from (2.6) that

$$E(E(\text{ess sup}_{\tau \in \mathcal{T}} |\hat{L}_\tau| \mathbf{1}_{(B_\varepsilon^s)^c} | \mathcal{F}_s)) = \sup_{\tau \in \mathcal{T}} E|\hat{L}_\tau| \mathbf{1}_{(B_\varepsilon^s)^c} \rightarrow 0 \quad (2.8)$$

as $s \rightarrow t^-$. Letting $s \rightarrow t^-$ in (2.7) and using (2.6), (2.8) yields

$$\limsup_{s \rightarrow t^-} U_s \leq \varepsilon + E(\hat{L}_t | \mathcal{F}_{t-}).$$

Since the above inequality holds for every $\varepsilon > 0$ and the filtration \mathcal{F} is quasi-left continuous, it follows that $\limsup_{s \rightarrow t^-} U_s \leq \hat{L}_t$, which when combined with (2.5) gives

$$L_{t-} \vee Y_t = \lim_{s \rightarrow t^-} U_s.$$

By this and (2.5),

$$Y_{t-} = \lim_{s \rightarrow t^-} Y_s \leq \lim_{s \rightarrow t^-} U_s = L_{t-} \vee Y_t,$$

and the proof is complete. \square

Example 2.4. The conclusion of Proposition 2.3 does not hold if the quasi-continuity of filtration is omitted from the hypotheses. To see this let us consider a random variable ξ such that $P(\xi = 5) = P(\xi = 1) = 1/2$. Let $T > 1$, $\mathcal{G} = \sigma(\xi)$. Define $\mathcal{F} = \{\mathcal{F}_t, t \in [0, T]\}$ as follows: $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t \in [0, 1)$ and $\mathcal{F}_t = \mathcal{G}$ for $t \in [1, T]$. Then the filtration \mathcal{F} is right-continuous but is not quasi-left continuous. Let $V = 0$ and $L_t = 2$ for $t \in [0, 1)$, $L_t = 0$ for $t \in [1, T]$. Since $Y_t^n \geq Y_t^0 \geq L_t$ for $t \in [0, T]$, it follows that

$$Y_t^n = Y_t, \quad t \in [0, T],$$

where Y is defined by (2.2) and Y_t^n is a solution of (2.10) with \mathcal{F}, ξ, L defined above and $f = V^n = 0$. Moreover,

$$Y_t = \begin{cases} E\xi, & t \in [0, 1), \\ \xi, & t \in [1, T], \end{cases}$$

from which it follows that $Y_{1-} \neq L_{1-} \vee Y_1$.

Remark 2.5. That Proposition 2.3 is not true if we drop the assumption of quasi-left continuity of filtration stems from the fact that the jumps of Y in predictable times can be produced by its martingale part.

Our Proposition 2.3 follows from a more general Proposition 2.6 proved in [4]. We have decided to provide Proposition 2.3 here to make our presentation self-contained in the important case of quasi-left continuous filtration. The second reason is that the proof of Proposition 2.3 is much simpler than that of Proposition 2.6.

Proposition 2.6. *Under the assumptions of Proposition 2.3 but with \mathcal{F} only satisfying the usual conditions,*

$$Y_{t-} = L_{t-} \vee ({}^pY_t + {}^pV_t - V_{t-}), \quad t \in [0, T], \quad (2.9)$$

where pY (resp. pV) denotes the predictable projection of the process Y (resp. V).

Proof. See [4, Proposition 2.34]. □

In the rest of this section (Y^n, M^n) stands for the solution of the BSDE

$$Y_t^n = \xi + \int_t^T f(r, Y_r^n) dr + \int_t^T dK_r^n + \int_t^T dV_r^n - \int_t^T dM_r^n, \quad t \in [0, T], \quad (2.10)$$

where

$$K_t^n = \int_0^t n(Y_r^n - L_r)^- dr, \quad t \in [0, T]$$

and $\{V^n\} \subset \mathcal{V}_0$ are processes such that $dV^n \leq dV^{n+1}$, $n \geq 1$, and $V_t^n \rightarrow V_t$, $t \in [0, T]$.

Definition. We say that a pair (Y, M) is a supersolution of BSDE($\xi, f + dV$) if there exists a process $C \in \mathcal{V}_0^{1,+}$ such that (Y, M) is a solution of BSDE($\xi, f + dV + dC$).

Let us consider the following hypotheses.

(H2) $[0, T] \ni t \mapsto f(t, y) \in L^1(0, T)$ for every $y \in \mathbb{R}$,

(H3) $\mathbb{R} \ni y \mapsto f(t, y)$ is continuous for a.e. $t \in [0, T]$.

Lemma 2.7. *Assume that $Y_t^n \nearrow Y_t$, $t \in [0, T]$, Y is a càdlàg process of class (D), $f_Y \equiv f(\cdot, Y) \in L^1(\mathcal{F})$, $V \in \mathcal{V}_0^1$ and (H1)–(H3) are satisfied. Then Y is the smallest supersolution of $\text{BSDE}(\xi, f_Y + dV)$ such that $Y_t \geq L_t$ for a.e. $t \in [0, T]$.*

Proof. Put

$$\tilde{Y}_t^n = Y_t^n + \int_0^t f(r, Y_r^n) dr + \int_0^t dV_r^n, \quad t \in [0, T].$$

Observe that \tilde{Y}^n is a supermartingale of class (D) and that by (H1)–(H3), $\tilde{Y}_t^n \rightarrow \tilde{Y}_t$, $t \in [0, T]$, where

$$\tilde{Y}_t = Y_t + \int_0^t f(r, Y_r) dr + \int_0^t dV_r, \quad t \in [0, T].$$

Since \tilde{Y} is also a supermartingale of class (D), there exist $M \in \mathcal{M}_0$ and $K \in \mathcal{V}_0^{1,+}$ such that

$$\tilde{Y}_t = \tilde{Y}_0 + K_t + M_t, \quad t \in [0, T].$$

Hence

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T dK_r - \int_t^T dM_r, \quad t \in [0, T],$$

i.e. (Y, M) is a supersolution of $\text{BSDE}(\xi, f_Y + dV)$. Let (\hat{Y}, \hat{M}) be a supersolution of $\text{BSDE}(\xi, f_Y + dV)$ such that $L_t \leq \hat{Y}_t$ for a.e. $t \in [0, T]$. By the definition of a supersolution there exists $C \in \mathcal{V}_0^{1,+}$ such that

$$\hat{Y}_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T dC_r - \int_t^T d\hat{M}_r, \quad t \in [0, T].$$

Let (\bar{Y}^n, \bar{M}^n) be a solution of the BSDE

$$\bar{Y}_t^n = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T n(\bar{Y}_r^n - L_r)^- dr - \int_t^T d\bar{M}_r^n, \quad t \in [0, T].$$

Since

$$\begin{aligned} Y_t &= \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T n(Y_r - L_r)^- dr + \int_t^T dV_r \\ &\quad + \int_t^T dK_r - \int_t^T dM_r, \quad t \in [0, T], \end{aligned}$$

it follows from [13, Proposition 2.1] that $\bar{Y}_t^n \leq Y_t$, $t \in [0, T]$. Arguing as in the case of the sequence $\{Y^n\}$ we show that there exist $\bar{K} \in \mathcal{V}_0^{1,+}$ and a càdlàg process \bar{Y} of class (D) (since $\bar{Y}^n \leq Y$) such that

$$\bar{Y}_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T d\bar{K}_r + \int_t^T dV_r - \int_t^T d\bar{M}_r, \quad t \in [0, T]$$

and $\bar{Y}_t^n \nearrow \bar{Y}_t$ for every $t \in [0, T]$. By [13, Proposition 2.1], $\bar{Y}_t^n \leq \hat{Y}_t$, $t \in [0, T]$, because

$$\hat{Y}_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T n(\hat{Y}_r - L_r)^- dr + \int_t^T dC_r - \int_t^T d\hat{M}_r, \quad t \in [0, T].$$

Thus

$$\bar{Y}_t \leq \hat{Y}_t, \quad t \in [0, T]. \quad (2.11)$$

Now we will show that $\bar{Y}_t = Y_t$, $t \in [0, T]$. To this end, let us set

$$\bar{K}_t^n = \int_0^t n(\bar{Y}_r^n - L_r)^- dr, \quad U_{t-}^n = \widehat{\text{sgn}}(Y_{t-}^n - \bar{Y}_{t-}^n),$$

where $\hat{x} = \frac{x}{|x|} \mathbf{1}_{\{x \neq 0\}}$. By the Tanaka-Meyer formula, for every $\tau \in \mathcal{T}$ we have

$$\begin{aligned} |Y_t^n - \bar{Y}_t^n| &\leq |Y_\tau^n - \bar{Y}_\tau^n| + \int_t^\tau U_{r-}^n d(K^n - \bar{K}^n)_r + \int_t^\tau U_{r-}^n (f(r, Y_r^n) - f(r, Y_r)) dr \\ &\quad - \int_t^\tau U_{r-}^n d(M^n - \bar{M}^n)_r \leq |Y_\tau^n - \bar{Y}_\tau^n| + \int_t^\tau |f(r, Y_r^n) - f(r, Y_r)| dr \\ &\quad - \int_t^\tau U_{r-}^n d(M^n - \bar{M}^n)_r, \quad t \in [0, \tau]. \end{aligned}$$

Let $\{\tau_k\} \subset \mathcal{T}$ be a fundamental sequence for $M^n - \bar{M}^n$. Since $Y^n - \bar{Y}^n$ is of class (D), replacing τ by τ_k in the above inequality, taking the expectations and then letting $k \rightarrow \infty$ we get

$$E|Y_t^n - \bar{Y}_t^n| \leq E \int_{t \wedge \tau}^\tau |f(r, Y_r^n) - f(r, Y_r)| dr, \quad t \in [0, T]. \quad (2.12)$$

By (H1),

$$f(r, Y_r) - \mu Y_r \leq f(r, Y_r^n) - \mu Y_r^n \leq f(r, Y_r^1) - \mu Y_r^1,$$

so applying the Lebesgue dominated convergence theorem shows that the right-hand side of (2.12) tends to zero. Therefore $\bar{Y}_t = Y_t$, $t \in [0, T]$, which when combined with (2.11) implies that $Y_t \leq \hat{Y}_t$, $t \in [0, T]$. \square

Corollary 2.8. *Under the assumptions of Lemma 2.7,*

$$Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} E \left(\int_t^\tau f(r, Y_r) dr + \int_t^\tau dV_r + \hat{L}_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau < T\}} | \mathcal{F}_t \right)$$

for every càdlàg process \hat{L} such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T]$.

Proof. From Lemma 2.7 it is clear that the process Y is the smallest supersolution of BSDE $(\xi, f_Y + dV)$ such that $Y_t \geq \hat{L}_t$, $t \in [0, T]$. From this we conclude that \tilde{Y} defined in the proof of Lemma 2.7 is the smallest supermartingale with the property that $\tilde{Y}_T = \tilde{\xi} \equiv \xi + \int_0^T f(r, Y_r) dr + \int_0^T dV_r$ majorizing the process $\tilde{L}_t = \hat{L}_t + \int_0^t f(r, Y_r) dr + \int_0^t dV_r$. Therefore

$$\tilde{Y}_t = \text{ess sup}_{\tau \in \mathcal{T}_t} E(\tilde{L}_\tau \mathbf{1}_{\{\tau < T\}} + \tilde{\xi} \mathbf{1}_{\{\tau = T\}} | \mathcal{F}_t),$$

from which the desired result immediately follows. \square

Corollary 2.9. *Under the assumptions of Lemma 2.7,*

$$Y_{t-} = \hat{L}_{t-} \vee ({}^pY_t + {}^pV_t - V_{t-}), \quad t \in [0, T].$$

for any càdlàg process \hat{L} such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T]$.

Proof. Follows immediately from Corollary 2.8 and Proposition 2.6. \square

Lemma 2.10. *Assume that Y is a supermartingale of class \mathcal{S}^2 admitting the decomposition*

$$Y_t = Y_0 - K_t + M_t, \quad t \in [0, T]$$

for some $K \in {}^p\mathcal{V}_0^+$, $M \in \mathcal{M}_0$. Then there exists $c > 0$ (not depending on Y) such that

$$E[Y]_T + E|K_T|^2 \leq cE \sup_{t \leq T} |Y_t|^2.$$

Proof. Without loss of generality we may assume that Y is bounded from above, for otherwise we can first prove the inequality for the supermartingale $Y \wedge n$ and then pass with n to the limit. By the Itô-Meyer formula, for every $\tau \in \mathcal{T}$,

$$|Y_t|^2 = |Y_\tau|^2 + \int_{t \wedge \tau}^\tau Y_{r-} dK_r - \int_{t \wedge \tau}^\tau Y_{r-} dM_r - \int_{t \wedge \tau}^\tau d[Y]_r, \quad t \in [0, \tau]. \quad (2.13)$$

We also have

$$Y_t = Y_\tau + \int_{t \wedge \tau}^\tau dK_r - \int_{t \wedge \tau}^\tau dM_r, \quad t \in [0, \tau]. \quad (2.14)$$

Let $\{\tau_k\} \subset \mathcal{T}$ be a fundamental sequence for M . Then replacing τ by τ^k in the above equation, taking the expectation and then letting $k \rightarrow \infty$ we get

$$EK_T = EY_0 - EY_T < \infty. \quad (2.15)$$

Using the localization procedure one can deduce from (2.13) and (2.15) that

$$EY_t^2 + E \int_t^T d[Y]_r \leq E|Y_T|^2 + \|Y\|_\infty EK_T < \infty.$$

Let us recall that since K is predictable $E[Y]_T = E[M]_T + E[K]_T$. Squaring both sides of (2.14), applying Doob's L^2 -inequality and performing standard calculations we conclude that there exists $c_1 > 0$ such that

$$E|K_T|^2 \leq c_1(E \sup_{t \leq T} |Y_t|^2 + E[Y]_T). \quad (2.16)$$

By (2.13) and Doob's L^2 -inequality,

$$E \sup_{t \leq T} |Y_t|^2 + E[Y]_T \leq E \sup_{t \leq T} |Y_t|^2 + \alpha E \sup_{t \leq T} |Y_t|^2 + \frac{1}{\alpha} E|K_T|^2 + 4\alpha E \sup_{t \leq T} |Y_t|^2 + \frac{1}{\alpha} E[Y]_T$$

for every $\alpha > 0$. The lemma follows from (2.16) and the above inequality with $\alpha = 2c_1 + 2$. \square

Lemma 2.11. *Assume that*

$$Y_t^n = Y_0^n - A_t^n + M_t^n, \quad t \in [0, T], \quad (2.17)$$

where $A^n \in {}^p\mathcal{V}_0^2$, $M^n \in \mathcal{M}_0^2$, $Y^1, Y \in \mathcal{S}^2$ and $Y_t^n \leq Y_t^{n+1}$, $t \in [0, T]$, $n \in \mathbb{N}$. Then the process Y defined as $Y_t = \lim_{n \rightarrow \infty} Y_t^n$, $t \in [0, T]$, is càdlàg.

Proof. By Lemma 2.10,

$$E|A^n|_T^2 + E[M^n]_T \leq cE \sup_{t \in [0, T]} |Y_t^n|^2 \leq cE \sup_{t \in [0, T]} (|Y_t^1|^2 \vee |Y_t|^2).$$

It follows in particular that $\sup_n E|M_T^n|^2 < \infty$. Therefore there is $X \in L^2(\mathcal{F}_T)$ such that $M_T^n \rightarrow X$ weakly in $L^2(\mathcal{F}_T)$. Let N be a càdlàg version of the martingale $E(X|\mathcal{F}_t)$, $t \in [0, T]$. Then for every $\tau \in \mathcal{T}$, $M_\tau^n \rightarrow N_\tau$ weakly in $L^2(\mathcal{F}_T)$. Indeed, for any $Z \in L^2(\mathcal{F}_T)$,

$$\begin{aligned} EM_\tau^n Z &= E(E(M_\tau^n|\mathcal{F}_\tau)Z) = E(M_\tau^n E(Z|\mathcal{F}_\tau)) \\ &\rightarrow E(XE(Z|\mathcal{F}_\tau)) = E(E(X|\mathcal{F}_\tau)Z) = EN_\tau Z. \end{aligned}$$

Put

$$A_t = Y_0 - Y_t + N_t, \quad t \in [0, T].$$

Then for every $\tau \in \mathcal{T}$,

$$A_\tau^n = Y_0^n - Y_\tau^n + M_\tau^n \rightarrow Y_0 - Y_\tau + N_\tau = A_\tau$$

weakly in $L^2(\mathcal{F}_T)$. From the above convergence we conclude that $A_\sigma \leq A_\tau$ for every $\sigma, \tau \in \mathcal{T}$ such that $\sigma \leq \tau$. From the section theorem it now follows that A is an increasing process. Consequently, Y is càdlàg by [20, Lemma 2.2]. \square

We will need the following hypotheses.

(H4) $\xi \in L^1(\mathcal{F}_T)$, $V \in \mathcal{V}_0^1$, $f(\cdot, 0) \in L^1(\mathcal{F})$.

(H5) There exists $X \in \mathcal{V}^1 \oplus \mathcal{M}$ such that $X_t \geq L_t$ for a.e. $t \in [0, T]$ and $f^-(\cdot, X) \in L^1(\mathcal{F})$.

Let $s \in [0, T)$ and let $\tau \in \mathcal{T}_t$. We will say that a sequence of processes $\{X^n\}$ converges to X uniformly in probability on $[s, \tau)$ (ucp on $[s, \tau)$ for short) if for every subsequence $\{n'\}$ there is a further subsequence $\{n''\}$ such that $X^{n''} \rightarrow X$ a.s. uniformly on compact subsets of $[s, \tau)$.

Theorem 2.12. *Assume that (H1)–(H4) hold. Then there exists a solution (Y, M, K) of RBSDE $(\xi, f + dV, L)$ such that $K \in \mathcal{V}^{1,+}$ iff (H5) is satisfied. Moreover, under (H1)–(H5), $Y_t^n \nearrow Y_t$, $t \in [0, T]$, $Y^n \rightarrow Y$ and $\int_s^\cdot dK_r^n \rightarrow \int_s^\cdot dK_r$ in ucp on $[s, \tau_s)$ for every $s \in [0, T)$, where $\tau_s = \inf\{t > s; \Delta K_t > 0\}$, and finally, for every $\tau \in \mathcal{T}$, $EK_\tau^n \rightarrow EK_\tau$.*

Proof. By [13, Lemma 2.3], if there exists a solution of RBSDE $(\xi, f + dV, L)$ such that $K \in \mathcal{V}^{1,+}$ then (H5) is satisfied with $X = Y$. Suppose now that (H5) is satisfied. By [13, Proposition 2.1], for every $n \geq 0$,

$$Y_t^n \leq Y_t^{n+1}, \quad t \in [0, T]. \quad (2.18)$$

By (H5) there exists X of class (D) such that

$$X_t \geq L_t \text{ for a.e. } t \in [0, T], \quad f^-(\cdot, X_\cdot) \in L^1(\mathcal{F}) \quad (2.19)$$

and

$$X_t = X_0 - U_t + N_t, \quad t \in [0, T]$$

for some $N \in \mathcal{M}_0$, $U \in \mathcal{V}_0^1$. Clearly,

$$\begin{aligned} X_t &= X_T + \int_t^T f(r, X_r) dr - \int_t^T f^+(r, X_r) + \int_t^T f^-(r, X_r) dr \\ &\quad + \int_t^T dU_r - \int_t^T dN_r, \quad t \in [0, T]. \end{aligned}$$

Let (\bar{X}, \bar{N}) be a solution of the BSDE

$$\begin{aligned} \bar{X}_t &= X_T \vee \xi + \int_t^T f(r, \bar{X}_r) dr + \int_t^T f^-(r, X_r) dr \\ &\quad + \int_t^T dV_r^+ \int_t^T dU_r^+ - \int_t^T d\bar{N}_r, \quad t \in [0, T]. \end{aligned}$$

Then by [13, Proposition 2.1], $\bar{X}_t \geq X_t$, $t \in [0, T]$, and hence, by (2.19), $\bar{X}_t \geq L_t$ for a.e. $t \in [0, T]$. Therefore

$$\begin{aligned} \bar{X}_t &= X_T \vee \xi + \int_t^T f(r, \bar{X}_r) dr + \int_t^T n(\bar{X}_r - L_r)^- dr \\ &\quad + \int_t^T f^-(r, X_r) dr \int_t^T dV_r^+ + \int_t^T dU_r^+ - \int_t^T d\bar{N}_r, \quad t \in [0, T], \end{aligned}$$

so using (2.18) and once again [13, Proposition 2.1] gives

$$Y_t^0 \leq Y_t^n \leq \bar{X}_t, \quad t \in [0, T]. \quad (2.20)$$

By [13, Lemma 2.3],

$$E \int_0^T |f(r, \bar{X}_r)| dr + E \int_0^T |f(r, Y_r^0)| dr < \infty. \quad (2.21)$$

Write $\varphi(x) = x/(1+x)$, $x \geq 0$ and $\bar{Y}_t^n \equiv Y_t^n - Y_t^0 + V_t^n - V_t^0$, $t \in [0, T]$, and observe that by (2.18), $\bar{Y}_t^n \geq 0$ for $t \in [0, T]$. By the Itô-Meyer formula,

$$\begin{aligned} \varphi(\bar{Y}_t^n) &= \varphi(\bar{Y}_0^n) + \int_0^t \varphi'(\bar{Y}_{r-}^n) d\bar{Y}_r^n + \frac{1}{2} \int_0^t \varphi''(\bar{Y}_{r-}^n) d[\bar{Y}^n]_r^c \\ &\quad + \sum_{0 \leq s \leq t} (\Delta \varphi(\bar{Y}_s^n) - \varphi'(\bar{Y}_{s-}^n) \Delta \bar{Y}_s^n), \quad t \in [0, T]. \end{aligned}$$

For $t \in [0, T]$ set

$$A_t^n = \int_0^t \varphi'(\bar{Y}_{r-}^n) dK_r^n, \quad B_t^n = -\frac{1}{2} \int_0^t \varphi''(\bar{Y}_{r-}^n) d[\bar{Y}^n]_r^c,$$

$$\begin{aligned}
C_t^n &= - \sum_{0 < s \leq t} (\Delta \varphi(\bar{Y}_s^n) - \varphi'(\bar{Y}_{s-}^n) \Delta \bar{Y}_s^n), \quad \tilde{C}_t^n = (C_t^n)^p, \\
F_t^n &= \int_0^t \varphi'(\bar{Y}_{r-}^n) (f(r, Y_r^n) - f(r, Y_r^0)) dr, \\
N_t^n &= \int_0^t \varphi'(\bar{Y}_r^n) d(M^n - M^0)_r + (\tilde{C}_t^n - C_t^n), \quad Z_t^n = \varphi(\bar{Y}_t^n) - L_t^n.
\end{aligned}$$

Then

$$Z_t^n = Z_0^n - A_t^n - B_t^n - \tilde{C}_t^n + N_t^n, \quad t \in [0, T].$$

Since φ is nondecreasing and concave, $A^n, B^n, C^n \in \mathcal{V}_0^+$. By (H1) and (2.18),

$$f(r, Y_r) - f(r, Y_r^0) - \mu(Y_r^n - Y_r^0) \leq f(r, Y_r^n) - f(r, Y_r^0) \leq \mu(Y_r^n - Y_r^0)$$

for a.e. $r \in [0, T]$. Hence

$$|f(r, Y_r^n) - f(r, Y_r^0)| \leq 2|\mu||Y_r - Y_r^0| + |f(r, Y_r) - f(r, Y_r^0)| \quad (2.22)$$

for a.e. $r \in [0, T]$. By (2.18), (2.20), (2.21) and (H3),

$$E \int_0^T |f(r, Y_r)| < \infty, \quad (2.23)$$

where

$$Y_t = \sup_{n \geq 0} Y_t^n, \quad t \in [0, T]. \quad (2.24)$$

Since φ' is bounded, it follows from (2.22) that there exists a stationary sequence $\{\sigma_k\} \in \mathcal{T}$ (i.e. $P(\liminf_{k \rightarrow \infty} \{\sigma_k = T\}) = 1$) such that $\sup_{n \geq 1} E|F^n|_{\sigma_k}^2 < \infty$. By Lemma 2.10 there exists $c > 0$ not depending of n such that

$$E|A_{\sigma_k}^n|^2 + E|B_{\sigma_k}^n|^2 + E|\tilde{C}_{\sigma_k}^n|^2 + E[Z^n]_{\sigma_k} \leq c.$$

By (2.18) and our assumptions on $\{V_n\}$, $Z_t^n \leq Z_t^{n+1}$, $t \in [0, T]$, $n \geq 0$, and

$$Z_t^n \nearrow Z_t, \quad t \in [0, T],$$

where $Z_t = \varphi(\bar{Y}_t) - F_t$, $\bar{Y}_t = Y_t + V_t - Y_t^0 - V_t^0$, and

$$F_t = \lim_{n \rightarrow \infty} F_t^n, \quad t \in [0, T]. \quad (2.25)$$

By (2.22)–(2.24) and (H3),

$$\int_0^t f(r, Y_r^n) dr \rightarrow \int_0^t f(r, Y_r) dr, \quad t \in [0, T]. \quad (2.26)$$

Since φ' is bounded and continuous, it follows from (2.26) that

$$F_t = \int_0^t \varphi'(\bar{Y}_r) (f(r, Y_r) - f(r, Y_r^0)) dr, \quad t \in [0, T]. \quad (2.27)$$

By Lemma 2.11, Z is a càdlàg process, so Y is càdlàg, too. Set

$$S_t = Y_t + \int_0^t f(r, Y_r) dr + \int_0^t dV_r, \quad t \in [0, T].$$

Observe that S^n defined as

$$S_t^n = Y_t^n + \int_0^t f(r, Y_r^n) dr + \int_0^t dV_r^n, \quad t \in [0, T]$$

is a supermartingale and by (2.24), (2.26) and the assumptions on $\{V^n\}$, $S_t^n \rightarrow S_t$, $t \in [0, T]$. From the last convergence and (2.20), (2.27) it follows that S is a càdlàg supermartingale of class (D). Therefore there exist $K \in {}^p\mathcal{V}_0^{1,+}$ and $M \in \mathcal{M}_0$ such that

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dK_r + \int_t^T dV_r - \int_t^T dM_r, \quad t \in [0, T]. \quad (2.28)$$

Let

$$R_t^n = Y_t^n + V_t^n, \quad R_t = Y_t + V_t, \quad t \in [0, T].$$

Then for $t \in [0, T]$,

$$R_t^n = R_0^n - \int_0^t f(r, Y_r^n) dr - K_t^n + M_t^n, \quad {}^pR_t^n = R_0^n - \int_0^t f(r, Y_r^n) dr - K_t^n + M_{t-}^n,$$

$$R_t = R_0 - \int_0^t f(r, Y_r) dr - K_t + M_t,$$

and since K is predictable,

$${}^pR_t = R_0 - \int_0^t f(r, Y_r) dr - K_t + M_{t-}.$$

Since $R_t^n \nearrow R_t$, $t \in [0, T]$, it follows that ${}^pR_t^n \nearrow {}^pR_t$, $t \in [0, T]$. Let us fix $s \in [0, T]$. Observe that

$${}^pR_t^n = R_{t-}^n, \quad t \in [0, T], \quad {}^pR_t = R_{t-}, \quad t \in (s, \tau_s).$$

By Dini's theorem, $R^n \rightarrow R$ in ucp on $[s, \tau_s]$. Since by the assumption $V^n \rightarrow V$ uniformly on $[0, T]$, $Y^n \rightarrow Y$ in ucp on $[s, \tau_s]$. Let

$$R_t^{s,n} = Y_t^{n,(s)} + V_t^{n,(s)}, \quad R_t^s = Y_t^{(s)} + V_t^{(s)}, \quad t \in [0, T]$$

with the notation $W_t^{(s)} = W_{t \vee s} - W_s$. Then

$$R_t^{s,n} = - \int_s^{t \vee s} f(r, Y_r^n) dr - \int_s^{t \vee s} dK_r^n + \int_s^{t \vee s} dM_r^n, \quad t \in [0, T]$$

and

$$R_t^s = - \int_s^{t \vee s} f(r, Y_r) dr - \int_s^{t \vee s} dK_r + \int_s^{t \vee s} dM_r, \quad t \in [0, T].$$

Let $\{T_m^s, m \geq 1\}$ be an announcing sequence for τ_s (τ_s is predictable since K is predictable). Then by what has already been proved,

$$\sup_{t \in [0, s \vee T_m^s]} |Y_t^{n,(s)} - Y_t^{(s)}| \rightarrow 0, \quad \sup_{t \in [0, s \vee T_m^s]} |R_t^{s,n} - R_t^s| \rightarrow 0. \quad (2.29)$$

By the assumptions on $\{V_n\}$ and (2.20),

$$\sup_{n \geq 1} EK_T^n < \infty. \quad (2.30)$$

Therefore by [18, Proposition 1-5] the sequence $\{R^{s,n} - R^s\}$ satisfies the condition UT. Therefore applying the results of [9] we obtain

$$[R^{s,n} - R^s]_{T_m^s \vee s} = [M^{n,(k)} - M^{(k)}]_{T_m^s \vee s} \rightarrow_P 0$$

as $n \rightarrow \infty$, or, equivalently,

$$\sup_{t \in [0, s \vee T_m^k]} |M_t^{n,(s)} - M_t^{(s)}| \rightarrow_P 0.$$

From this we conclude that for every $m \geq 1$,

$$\sup_{t \in [s, s \vee T_m^s]} \left| \int_s^t dK_r^n - \int_s^t dK_r \right| \rightarrow_P 0. \quad (2.31)$$

Observe that by (2.30), $Y_t \geq L_t$ for a.e. $t \in [0, T]$. Let \hat{L} be a càdlàg process such that $L_t \leq \hat{L}_t \leq Y_t$ for a.e. $t \in [0, T]$. Then

$$\int_s^{s \vee T_m^s} (Y_{t-}^n - \hat{L}_{t-}) dK_t^n \leq n \int_s^{s \vee T_m^s} (Y_t^n - \hat{L}_t)(Y_t^n - \hat{L}_t)^- dt \leq 0. \quad (2.32)$$

By (2.29) and (2.31),

$$\int_s^{s \vee T_m^s} (Y_t^n - \hat{L}_t) dK_t^n \rightarrow \int_s^{s \vee T_m^s} (Y_t - \hat{L}_t) dK_t = \int_s^{s \vee T_m^s} (Y_{t-} - \hat{L}_{t-}) dK_t$$

since $\Delta K_t = 0$ for $t \in (s, \tau_s)$. Since $Y_{t-} \geq \hat{L}_{t-}$ for $t \in (0, T]$, it follows from (2.32) that for every $s \in [0, T]$,

$$\int_s^{\tau_s} (Y_{t-} - \hat{L}_{t-}) dK_t = 0. \quad (2.33)$$

By the definition of $\{\tau_s\}$, $\Delta K_{\tau_s} > 0$ on $\{\tau_s < \infty\}$. Hence

$${}^p Y_{\tau_s} + {}^p V_{\tau_s} - V_{\tau_s-} < Y_{\tau_s-} \quad \text{on } \{\tau_s < \infty\}$$

since ${}^p Y_{\tau_s} - Y_{\tau_s-} = -\Delta K_{\tau_s} - ({}^p V_{\tau_s} - V_{\tau_s-})$. Therefore by Corollary 2.9, $Y_{\tau_s-} = \hat{L}_{\tau_s-}$ on $\{\tau_s < \infty\}$, which when combined with (2.33) shows that

$$\int_0^T (Y_{t-} - \hat{L}_{t-}) dK_t = 0.$$

Integrability of K follows from (2.27) and the fact that Y is of class (D). \square

Proposition 2.13. *Assume that $(\xi^i, f^i + dV^i, L)$, $i = 1, 2$, satisfy (H1)–(H5). Let (Y^i, M^i, K^i) be a solution of RBSDE $(\xi^i, f^i + dV^i, L)$, $i = 1, 2$. If $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$ and either $f^1(t, Y_t^1) \leq f^2(t, Y_t^2)$ or $f^1(t, Y_t^2) \leq f^2(t, Y_t^2)$ for a.e. $t \in [0, T]$ then $dK^2 \leq dK^1$.*

Proof. By [13, Proposition 2.1],

$$Y_t^{1,n} \leq Y_t^{2,n}, \quad t \in [0, T], \quad (2.34)$$

where $Y^{i,n}$ is a solution of (2.10) with (ξ, f, V) replaced by (ξ^i, f^i, V^i) , $i = 1, 2$. For $s \in [0, T)$ we set $\tau_s = \inf\{t > s; \Delta K_t^1 + \Delta K_t^2 > 0\}$. By Theorem 2.12,

$$\int_s^t dK_r^{n,i} \rightarrow \int_s^t dK^i, \quad t \in [s, \tau_s), \quad (2.35)$$

for every $s \in [0, T)$, where $K_t^{n,i} = \int_0^t n(Y_r^{n,i} - L_r)^- dr$. From this and (2.34) we conclude that for every $s \in [0, T)$,

$$dK^2 \leq dK^1 \quad \text{on } [s, \tau_s). \quad (2.36)$$

Let \hat{L} be a càdlàg process such that $L_t \leq \hat{L}_t \leq Y_t^i$ for a.e. $t \in [0, T]$, $i = 1, 2$ (for instance one can take $\hat{L} = Y^1 \wedge Y^2$). By the definition of a solution of $\text{RBSDE}(\xi^2, f^2 + dV^2, L)$, if $\Delta K_{\tau_s}^2 > 0$ then $Y_{\tau_s-}^2 = \hat{L}_{\tau_s-}$ on $\{\tau_s < \infty\}$. Therefore by the assumptions of the proposition and (2.34), if $\Delta K_{\tau_s}^2 > 0$ and $\tau_s < \infty$ then

$$\begin{aligned} \Delta K_{\tau_s}^1 &= -(^pY_{\tau_s}^1 - Y_{\tau_s-}^1) - (^pV_{\tau_s}^1 - V_{\tau_s-}^1) \geq -(^pY_{\tau_s}^1 - \hat{L}_{\tau_s-}) - (^pV_{\tau_s}^1 - V_{\tau_s-}^1) \\ &\geq -(^pY_{\tau_s}^2 - \hat{L}_{\tau_s-}) - (^pV_{\tau_s}^2 - V_{\tau_s-}^2) = -(^pY_{\tau_s}^2 - Y_{\tau_s-}^2) - (^pV_{\tau_s}^2 - V_{\tau_s-}^2) = \Delta K_{\tau_s}^2. \end{aligned}$$

This and (2.36) proves the proposition. \square

3 BSDEs with two reflecting barriers: the case $p = 1$

In this section we prove the existence of solutions of BSDEs with two reflecting barriers L, U which are separated by a semimartingale. In what follows the upper barrier U is a progressively measurable process such that $L_t \leq U_t$ for a.e. $t \in [0, T]$.

Definition. We say that a triple (Y, M, A) is a solution of the reflected BSDE with terminal condition ξ , right-hand side $f + dV$ and upper barrier U ($\text{RBSDE}(\xi, f + dV, U)$ for short) if

- (a) Y is an \mathcal{F} adapted càdlàg process of class (D), $M \in \mathcal{M}_{0,loc}$, $A \in \mathcal{V}_0^+$,
- (b) $Y_t \leq U_t$ for a.e. $t \in [0, T]$,
- (c) for every càdlàg process \hat{U} such that $U_t \geq \hat{U}_t \geq Y_t$ for a.e. $t \in [0, T]$,

$$\int_0^T (\hat{U}_{t-} - Y_{t-}) dA_t = 0,$$

- (d) $[0, T] \ni t \rightarrow f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r - \int_t^T dA_r - \int_t^T dM_r, \quad t \in [0, T].$$

From now on we adopt the convention that $\text{RBSDE}(\cdot, \cdot, L)$ denotes equation with lower barrier, while $\text{RBSDE}(\cdot, \cdot, U)$ denotes equation with upper barrier.

Definition. We say that a triple of processes (Y, M, R) is a solution of the reflected BSDE with terminal condition ξ , right-hand side $f + dV$, lower barrier L and upper barrier U (RBSDE($\xi, f + dV, L, U$) for short) if

- (a) Y is an \mathcal{F} -adapted càdlàg process and of class (D), $M \in \mathcal{M}_{0,loc}$, $R \in {}^p\mathcal{V}_0$,
- (b) $L_t \leq Y_t \leq U_t$ for a.e. $t \in [0, T]$,
- (c) for every càdlàg processes \hat{L}, \hat{U} such that $L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$ we have

$$\int_0^T (Y_{t-} - L_{t-}) dR_t^+ = \int_0^T (U_{t-} - Y_{t-}) dR_t^- = 0,$$

- (d) $[0, T] \ni t \mapsto f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = \xi + \int_t^T f(r, Y_r) dr + \int_t^T dV_r + \int_t^T dR_r - \int_t^T dM_r, \quad t \in [0, T].$$

Proposition 3.1. *Let (Y^i, M^i, R^i) be a solution of RBSDE($\xi^i, f^i + dV^i, L^i, U^i$), $i = 1, 2$. Assume that $\xi^1 \leq \xi^2$, $L_t^1 \leq L_t^2$, $U_t^1 \leq U_t^2$ for a.e. $t \in [0, T]$, $dV^1 \leq dV^2$, and either f^1 satisfies (H1) and $f^1(t, Y_t^2) \leq f^2(t, Y_t^2)$ for a.e. $t \in [0, T]$ or f^2 satisfies (H1) and $f^1(t, Y_t^1) \leq f^2(t, Y_t^1)$ for a.e. $t \in [0, T]$. Then $Y_t^1 \leq Y_t^2$, $t \in [0, T]$.*

Proof. The proof is analogous to the proof of Proposition 2.1. \square

Corollary 3.2. *Let assumption (H1) hold. Then there exists at most one solution of RBSDE($\xi, f + dV, L, U$).*

Let us consider the following hypothesis.

(H6) There exists $X \in \mathcal{V}^1 \oplus \mathcal{M}_{loc}$ such that

$$L_t \leq X_t \leq U_t \quad \text{for a.e. } t \in [0, T], \quad f(\cdot, X) \in L^1(\mathcal{F}).$$

In what follows $(Y^{n,m}, M^{n,m})$ is a solution of the BSDE

$$\begin{aligned} Y_t^{n,m} = & \xi + \int_t^T f(r, Y_r^{n,m}) dr + \int_t^T n(Y_r^{n,m} - L_r)^- dr \\ & - \int_t^T m(Y_r^{n,m} - U_r)^+ dr + \int_t^T dV_r - \int_t^T dM_r^{n,m}, \quad t \in [0, T] \end{aligned} \quad (3.1)$$

and $(\bar{Y}^n, \bar{M}^n, \bar{A}^n)$ is a solution of the RBSDE

$$\begin{aligned} \bar{Y}_t^n = & \xi + \int_t^T f(r, \bar{Y}_r^n) dr + \int_t^T n(\bar{Y}_r^n - L_r)^- dr \\ & - \int_t^T d\bar{A}_r^n + \int_t^T dV_r - \int_t^T d\bar{M}_r^n, \quad t \in [0, T] \end{aligned} \quad (3.2)$$

with upper barrier U . Write

$$\bar{K}_t^n = \int_0^t n(\bar{Y}_s^n - L_s)^- ds, \quad t \in [0, T].$$

Theorem 3.3. Assume (H1)–(H4). Then there exists a unique solution (Y, M, R) of $\text{RBSDE}(\xi, f + dV, L, U)$ such that $R \in \mathcal{V}_0^1$ iff (H6) is satisfied. Moreover, $\bar{Y}_t^n \nearrow Y_t$, $t \in [0, T]$, $d\bar{A}^n \leq d\bar{A}^{n+1}$, $n \in \mathbb{N}$, $\bar{A}_t^n \nearrow R_t^-$, $t \in [0, T]$, $\bar{Y}^n \rightarrow Y$, $\int_s^\cdot d\bar{K}_r^n \rightarrow \int_s^\cdot dR_r^+$ in ucp on $[s, \tau_s)$ for every $s \in [0, T)$, where $\tau_s = \inf\{t > s; \Delta R_t^+ > 0\}$, and $Y_t^{n,m} \rightarrow Y_t$, $t \in [0, T]$, as $n, m \rightarrow \infty$.

Proof. Assume that there exists a solution (Y, M, R) of $\text{RBSDE}(\xi, f + dV, L, U)$ such that $R \in \mathcal{V}_0^1$. Then $f(\cdot, Y) \in L^1(\mathcal{F})$ by [13, Lemma 2.3], so (H6) is satisfied with $X = Y$. Now assume (H6). By Theorem 2.12 there exists a unique solution of (3.2) such that $\bar{A}^n \in \mathcal{V}_0^1$. Moreover, by [13, Theorem 2.7], there exists a unique solution of (3.1), and by Theorem 2.12,

$$Y_t^{n,m} \searrow \bar{Y}_t^n, \quad t \in [0, T], \quad EA_T^{n,m} \rightarrow E\bar{A}_T^n, \quad (3.3)$$

where

$$A_t^{n,m} = \int_0^t m(Y_r^{n,m} - U_r)^+ dr, \quad t \in [0, T].$$

By [13, Proposition 2.1],

$$\hat{Y}_t \leq Y_t^{n,m} \leq \check{Y}_t, \quad t \in [0, T], \quad (3.4)$$

where $(\hat{Y}, \hat{M}, \hat{K})$ is a solution of $\text{RBSDE}(\xi, f + dV, U)$ and $(\check{Y}, \check{M}, \check{A})$ is a solution of $\text{RBSDE}(\xi, f + dV, L)$. By (H6) there exist $C \in \mathcal{V}_0^1$ and $N \in \mathcal{M}_{0,loc}$ such that

$$X_t = X_0 - C_t + N_t, \quad t \in [0, T].$$

Since $L_t \leq X_t \leq U_t$ for a.e. $t \in [0, T]$, we have

$$\begin{aligned} X_t &= X_T + \int_t^T f(r, X_r) dr + \int_t^T f^-(r, X_r) dr - \int_t^T f^+(r, X_r) dr \\ &\quad + \int_t^T dC_r + \int_t^T dV_r - \int_t^T m(X_r - U_r)^+ dr - \int_t^T dN_r, \quad t \in [0, T]. \end{aligned}$$

By [13, Theorem 2.7] there exist a solution (\bar{X}^m, \bar{N}^m) of BSDE

$$\begin{aligned} \bar{X}_t^m &= X_T \vee \xi + \int_t^T f(r, \bar{X}_r^m) dr + \int_t^T f^-(r, X_r) dr + \int_t^T dC_r \\ &\quad + \int_t^T dV_r - \int_t^T m(\bar{X}_r^m - U_r)^+ dr - \int_t^T d\bar{N}_r^m, \quad t \in [0, T]. \end{aligned}$$

By [13, Proposition 2.1], $\bar{X}_t^m \geq X_t$ for $t \in [0, T]$, so $\bar{X}_t^m \geq L_t$ for a.e. $t \in [0, T]$, which when combined with the above equation implies that

$$\begin{aligned} \bar{X}_t^m &= X_T \vee \xi + \int_t^T f(r, \bar{X}_r^m) dr + \int_t^T f^-(r, X_r) dr + \int_t^T dC_r + \int_t^T dV_r \\ &\quad + \int_t^T n(\bar{X}_t^m - L_r)^- dr - \int_t^T m(\bar{X}_r^m - U_r)^+ dr - \int_t^T d\bar{N}_r^m, \quad t \in [0, T]. \end{aligned}$$

Therefore applying once again [13, Proposition 2.1] we get

$$Y_t^{n,m} \leq \bar{X}_t^m, \quad t \in [0, T]. \quad (3.5)$$

Write

$$\bar{D}_t^m = \int_0^t m(\bar{X}_r^m - U_r)^+ dr, \quad t \in [0, T].$$

then by (3.5),

$$dA^{n,m} \leq d\bar{D}^m \quad \text{on } [0, T], \quad (3.6)$$

so by Theorem 2.12,

$$\bar{X}_t^m \searrow \bar{X}_t, \quad t \in [0, T], \quad E\bar{D}_T^m \rightarrow E\bar{D}_T, \quad (3.7)$$

where $(\bar{X}, \bar{N}, \bar{D})$ is a solution of RBSDE($X_T \vee \xi, f + f_X^- + dC^+ + dV, U$). From (3.7) and (3.3) we conclude that

$$E\bar{A}_T^n \leq E\bar{D}_T, \quad n \geq 1. \quad (3.8)$$

By Proposition 2.3, for every $n \geq 1$,

$$d\bar{A}^n \leq d\bar{A}^{n+1} \quad \text{on } [0, T].$$

Therefore by (3.8) and [20, Lemma 2.2] there exists $\bar{A} \in {}^p\mathcal{V}_0^1$ such that

$$|d\bar{A}^n - d\bar{A}|_{TV} \rightarrow 0, \quad (3.9)$$

where $|\cdot|_{TV}$ denotes the total variation norm on $[0, T]$. Consequently, by Theorem 2.12,

$$\bar{Y}_t^n \nearrow \bar{Y}_t, \quad t \in [0, T], \quad E\bar{K}_T^n \rightarrow E\bar{K}_T, \quad (3.10)$$

where $(\bar{Y}, \bar{M}, \bar{K})$ is a solution of RBSDE($\xi, f + dV + d\bar{A}, L$). In particular, $\bar{Y}_t \geq L_t$ for a.e. $t \in [0, T]$ and for every càdlàg process \hat{L} such that $L_t \leq \hat{L}_t \leq \bar{Y}_t$ for a.e. $t \in [0, T]$,

$$\int_0^T (\bar{Y}_{t-} - \hat{L}_{t-}) d\bar{K}_t = 0.$$

On the other hand, since $(\bar{Y}^n, \bar{M}^n, \bar{A}^n)$ is a solution of (3.2), $\bar{Y}_t^n \leq U_t$ for a.e. $t \in [0, T]$ and for every càdlàg process \hat{U} such that $\bar{Y}_t^n \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$,

$$\int_0^T (\hat{U}_{t-} - \bar{Y}_{t-}^n) d\bar{A}_t^n = 0. \quad (3.11)$$

Let \hat{U} be a càdlàg process such that $\bar{Y}_t \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$ (let us recall that $\bar{Y}_t^n \leq \bar{Y}_t$, $t \in [0, T]$). Let $\sigma_s = \inf\{t > s; \Delta\bar{A}_t > 0\}$ and let $\{S_p^s, p \geq 1\}$ be an announcing sequence for σ_s . Then $\Delta\bar{A}_{\sigma_s} > 0$ on $\{\sigma_s < \infty\}$. We may assume that this inequality holds for every $\omega \in \{\sigma_s < \infty\}$, (3.9) holds for every $\omega \in \Omega$ and that (3.11) holds for every $\omega \in \Omega$ and $n \in \mathbb{N}$. Therefore, thanks to (3.9), for every $\omega \in \{\sigma_s < \infty\}$ there exists $n_0(\omega) \in \mathbb{N}$ such that $(\Delta\bar{A}_{\sigma_s}^n)(\omega) > 0$ for $n \geq n_0(\omega)$. From this and (3.11) we conclude that $\bar{Y}_{\sigma_s-}^n(\omega) = \hat{U}_{\sigma_s-}(\omega)$, $n \geq n_0(\omega)$, on $\{\sigma_s < \infty\}$. Since $\bar{Y}_t^n \leq \hat{U}_t$ for $t \in [0, T]$ and $\{\bar{Y}^n\}$ is increasing, $\bar{Y}_{\sigma_s-}^n(\omega) = \bar{Y}_{\sigma_s-}(\omega) = \hat{U}_{\sigma_s-}(\omega)$ for $n \geq n_0(\omega)$ on $\{\sigma_s < \infty\}$. Therefore for every $s \in [0, T]$ we have

$$(\hat{U}_{\sigma_s-} - Y_{\sigma_s-})\Delta\bar{A}_{\sigma_s} = 0 \quad \text{on } \{\sigma_s < \infty\}. \quad (3.12)$$

By (3.9)–(3.11),

$$0 = \int_s^{s \vee S_p^s} (\hat{U}_{t-} - \bar{Y}_{t-}^n) d\bar{A}_t^n = \int_s^{s \vee S_p^s} (\hat{U}_t - \bar{Y}_t^n) d\bar{A}_t^n \rightarrow \int_s^{s \vee S_p^s} (\hat{U}_t - \bar{Y}_t) d\bar{A}_t.$$

From this and (3.12) we get

$$\int_0^T (\hat{U}_{t-} - \bar{Y}_{t-}) d\bar{A}_t = 0. \quad (3.13)$$

Thus the triple $(\bar{Y}, \bar{M}, \bar{R})$, where $\bar{R} = \bar{K} - \bar{A}$, is a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. Now instead of (3.2) let us consider a solution $(\underline{Y}^m, \underline{M}^m, \underline{K}^m)$ of the RBSDE

$$\begin{aligned} \underline{Y}_t^m &= \xi + \int_t^T f(r, \underline{X}_r^m) dr + \int_t^T d\underline{K}_r^m \\ &\quad - \int_t^T m(\underline{Y}_r^m - U_r)^+ dr + \int_t^T dV_r - \int_t^T d\underline{M}_r^m, \quad t \in [0, T] \end{aligned}$$

with lower barrier L . Repeating, with some obvious changes, the proofs of the first parts of (3.3), (3.10) we show that

$$\underline{Y}_t^m \nearrow \underline{Y}_t, \quad Y_t^{n,m} \searrow \underline{Y}_t^m, \quad t \in [0, T], \quad (3.14)$$

where $(\underline{Y}, \underline{M}, \underline{R})$ is a solution of $\text{RBSDE}(\xi, f + dV, L, U)$ constructed analogously to (Y, M, R) . By Proposition 3.1, $(Y, M, R) = (\underline{Y}, \underline{M}, \underline{R}) \equiv (Y, M, R)$, so that by (3.3) and (3.14),

$$\bar{Y}_t^n \leq Y_t^{n,m} \leq \underline{Y}_t^m, \quad t \in [0, T]$$

and $Y_t^{n,m} \rightarrow Y_t$, $t \in [0, T]$, as $n, m \rightarrow \infty$. We now show that $\bar{A} = R^-$, $\bar{K} = R^+$. Observe that the triple (Y, M, R^-) is a solution of $\text{RBSDE}(\xi, f + dV + dR^+, U)$. Therefore by Theorem 2.12,

$$\tilde{Y}_t^m \searrow Y_t, \quad t \in [0, T], \quad \sup_{t \in [s, s \vee S_p^s]} \left| \int_s^t d\tilde{A}_r^m - \int_s^t dR_r^- \right| \rightarrow_P 0, \quad (3.15)$$

where the pair $(\tilde{Y}^n, \tilde{M}^n)$ is a solution of the BSDE

$$\tilde{Y}_t^m = \xi + \int_t^T f(r, \tilde{Y}_r^m) dr + \int_t^T dR_r^+ - \int_t^T d\tilde{A}_r^m + \int_t^T dV_r - \int_t^T d\tilde{M}_r^m, \quad t \in [0, T]$$

and $\tilde{A}_t^m = \int_0^t m(\tilde{Y}_r^m - U_r)^+ dr$, $t \in [0, T]$. By (3.15), $\tilde{Y}_t^m \geq L_t$ for a.e. $t \in [0, T]$. Therefore

$$\begin{aligned} \tilde{Y}_t^m &= \xi + \int_t^T f(r, \tilde{Y}_r^m) dr + \int_t^T n(\tilde{Y}_r^m - L_r)^- dr \\ &\quad + \int_t^T dR_r^+ - \int_t^T d\tilde{A}_r^m + \int_t^T dV_r - \int_t^T d\tilde{M}_r^m, \quad t \in [0, T] \end{aligned}$$

and hence, by [13, Proposition 2.1], $\tilde{Y}_t^m \geq Y_t^{n,m}$, $t \in [0, T]$. Consequently,

$$dA^{n,m} \leq d\tilde{A}^m \quad \text{on } [0, T] \quad (3.16)$$

for $n, m \in \mathbb{N}$. Since $d\bar{A}^m \leq d\bar{A}$, Theorem 2.12 implies that

$$\sup_{t \in [s, s \vee S_p^s]} \left| \int_s^t dA_r^{n,m} - \int_s^t d\bar{A}_r^n \right| \rightarrow_P 0$$

as $m \rightarrow \infty$. This when combined with (3.9) and (3.15), (3.16) shows that for every $s \in [0, T]$,

$$d\bar{A} \leq dR^- \quad \text{on } (s, \sigma_s). \quad (3.17)$$

In the reasoning preceding (3.12) we have showed that $\bar{Y}_{\sigma_k-}^n(\omega) = \hat{U}_{\sigma_k-}(\omega)$ for $n \geq n_0(\omega)$ on $\{\sigma_s < \infty\}$. This implies that ${}^p\bar{Y}_{\sigma_s-}^n - \bar{Y}_{\sigma_s-}^n \rightarrow {}^pY_{\sigma_s-} - Y_{\sigma_s-}$ on $\{\sigma_s < \infty\}$. Also $\Delta \bar{A}_{\sigma_s}^n \rightarrow \Delta \bar{A}_{\sigma_s}$ on $\{\sigma_s < \infty\}$ by (3.9). Therefore $\Delta \bar{K}_{\sigma_s} = 0$ on $\{\sigma_s < \infty\}$. Since $dR^+ \leq d\bar{K}$ by the minimality of the Jordan decomposition of dR , we have $\Delta R_{\sigma_s}^+ = 0$ on $\{\sigma_s < \infty\}$. Hence $\Delta R_{\sigma_s}^- = \Delta \bar{A}_{\sigma_s}$ on $\{\sigma_s < \infty\}$. This and (3.17) show that

$$d\bar{A} \leq dR^-.$$

In much the same way one can show that $d\bar{K} \leq dR^+$. Therefore by the minimality of the Jordan decomposition of the measure dR , $dR^- = d\bar{A}$ and $dR^+ = d\bar{K}$. \square

Let L, U be càdlàg and let (Y, M, R) be a solution of RBSDE($\xi, f + dV, L, U$) such that $R \in \mathcal{V}_0^1$. For $t \in [0, T]$ set

$$\mathcal{T}_t = \{\tau \in \mathcal{T}; \quad t \leq \tau \leq T\}$$

and consider the payoff function

$$R_t(\sigma, \tau) = \int_t^{\sigma \wedge \tau} f(r, Y_r) dr + \int_t^{\sigma \wedge \tau} dV_r + L_\tau \mathbf{1}_{\{\tau < T, \tau \leq \sigma\}} + U_\sigma \mathbf{1}_{\{\sigma < \tau\}} + \xi \mathbf{1}_{\{\sigma \wedge \tau = T\}}. \quad (3.18)$$

The lower \underline{W} and upper \overline{W} values of the stochastic game corresponding to $R_t(\cdot, \cdot)$ are defined as

$$\underline{W}_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \text{ess inf}_{\sigma \in \mathcal{T}_t} E(R_t(\sigma, \tau) | \mathcal{F}_t), \quad \overline{W}_t = \text{ess sup}_{\sigma \in \mathcal{T}_t} \text{ess inf}_{\tau \in \mathcal{T}_t} E(R_t(\sigma, \tau) | \mathcal{F}_t).$$

The game is said to have a value if $\underline{W}_t = \overline{W}_t$, $t \in [0, T]$.

Proposition 3.4. *Let (Y, M, R) be a solution of RBSDE($\xi, f + dV, L, U$). Then the stochastic game associated with payoff (3.18) has the value equal to Y , i.e.*

$$Y_t = \overline{W}_t = \underline{W}_t, \quad t \in [0, T].$$

Proof. It is enough to repeat step by step the proof of [16, Proposition 3.1]. \square

4 Nonintegrable solutions of reflected BSDEs

In Sections 2 and 3 under the assumption that (H1)–(H4) are satisfied necessary and sufficient conditions for the existence of a solution of reflected BSDE with one and two barriers are formulated. In the case of one barrier the necessary and sufficient condition (H5) relates the growth of the barrier L to the generator f . In the case of two barriers the corresponding condition (H6) consists of two parts. The first one, as in the case of

one barrier, relates the growth of the lower barrier L and upper barrier U to f . The second one, known as Mokobodski's condition, amounts to saying that there is some semimartingale between L and U . The question arises whether the solution still exists if we get rid of the conditions relating the growth of the barriers to f and we only impose minimal integrability conditions on L, U ensuring Snell envelope representation of a possible solution, i.e. ensuring that if a solution exists, it is of class (D). In the case of Brownian filtration and continuous barriers the question was investigated in [10]. It appears that the answer is positive but in general the reflecting process may be nonintegrable for every $q > 0$.

Theorem 4.1. *Assume that (H1)–(H4) are satisfied and L is of class (D). Then there exists a unique solution (Y, M, K) of $\text{RBSDE}(\xi, f + dV, L)$. Moreover, $Y_t^n \nearrow Y_t$, $t \in [0, T]$, $Y^n \rightarrow Y$, $\int_s dK_r^n \rightarrow \int_s dK_r$ in ucp on $[s, \tau_s)$ for every $s \in [0, T]$, where $\tau_s = \inf\{t > s; \Delta K_t > 0\}$.*

Proof. Let $(\tilde{Y}^n, \tilde{M}^n)$ be a solution of the BSDE

$$\tilde{Y}_t^n = \xi + \int_t^T f^+(r, \tilde{Y}_r^n) dr + \int_t^T n(\tilde{Y}_r^n - L_r)^- dr + \int_t^T dV_r - \int_t^T d\tilde{M}_r^n, \quad t \in [0, T].$$

Let X be a supermartingale of class (D) majorizing L (for instance we may take the solution of $\text{RBSDE}(\xi, 0, L)$ as X). Then the data (ξ, f^+, V, L) satisfy (H1)–(H4) and (H5) with X chosen above. Therefore by Theorem 2.12,

$$\tilde{Y}_t^n \nearrow \tilde{Y}_t, \quad t \in [0, T],$$

where $(\tilde{Y}, \tilde{M}, \tilde{K})$ is a solution of $\text{RBSDE}(\xi, f^+ + dV, L)$ such that $\tilde{K} \in \mathcal{V}_0^1$. By [13, Proposition 2.1], $\{Y^n\}$ is nondecreasing and

$$Y_t^0 \leq Y_t^n \leq \tilde{Y}_t^n \leq \tilde{Y}_t, \quad t \in [0, T]. \quad (4.1)$$

Put

$$Y_t = \sup_{n \geq 1} Y_t^n, \quad t \in [0, T]$$

and $\xi_n^k = Y_{\delta_k}^n$, $\xi^k = Y_{\delta_k}$, where

$$\delta_k = \inf\{t \geq 0; \int_0^t f^-(r, \tilde{Y}_r) dr \geq k\} \wedge T.$$

By (4.1), Y is of class (D), so $\xi_n^k \nearrow \xi^k$ a.s and in L^1 for each $k \geq 1$. Observe that the data $(\xi_n^k, f + dV, L)$ satisfy (H1)–(H6) on $[0, \delta_k]$ with $X = \tilde{Y}$. Let $(Y^{n,(k)}, M^{n,(k)})$ be a solution of (2.10) on $[0, \delta_k]$ with terminal condition ξ_n^k . It is clear that $(Y_t^{n,(k)}, M_t^{n,(k)}) = (Y_t^n, M_t^n)$, $t \in [0, \delta_k]$. By Theorem 2.12, $(Y^{n,(k)}, M^{n,(k)}, K^{n,(k)}) \rightarrow (Y^{(k)}, M^{(k)}, K^{(k)})$ on $[0, \delta_k]$, where $(Y^{(k)}, M^{(k)}, K^{(k)})$ is a solution of $\text{RBSDE}(\xi^k, f + dV, L)$ on $[0, \delta_k]$ and $K_t^{n,(k)} = \int_0^t n(Y_r^{n,(k)} - L_r)^- dr$, $t \in [0, \delta_k]$. Of course, $Y_t^{(k)} = Y_t$, $t \in [0, \delta_k]$. Observe that $\{\delta_k\}$ is stationary, i.e.

$$P(\liminf_{k \rightarrow \infty} \{\delta_k = T\}) = 1.$$

Also observe that from the fact that $Y_t^{(k)} = Y_t^{(k+1)}$ for $t \in [0, \delta_k]$ and the uniqueness of the Doob-Meyer decomposition it follows that

$$(Y_t^{(k+1)}, M_t^{(k+1)}, K_t^{(k+1)}) = (Y_t^{(k)}, M_t^{(k)}, K_t^{(k)}), \quad t \in [0, \delta_k]$$

for every $k \geq 1$. Therefore we may define process M, K on $[0, T]$ by putting

$$M_t = M_t^{(k)}, \quad K_t = K_t^{(k)}, \quad t \in [0, \delta_k].$$

It is clear that the triple (Y, M, K) is a solution of $\text{RBSDE}(\xi, f + dV, K)$. \square

The following hypothesis called the Mokobodski condition in the literature.

(H7) There exists a process $X \in \mathcal{V}^1 \oplus \mathcal{M}_{loc}$ such that

$$L_t \leq X_t \leq U_t \quad \text{for a.e. } t \in [0, T].$$

Theorem 4.2. *Assume (H1)–(H4), (H7). Then there exists a unique solution (Y, M, R) of $\text{RBSDE}(\xi, f + dV, L, U)$. Moreover, $\bar{Y}_t^n \nearrow Y_t$, $Y_t^{n,m} \rightarrow Y_t$, $t \in [0, T]$, $\bar{A}_t^n \nearrow R_t^-$, $\bar{Y}^n \rightarrow Y$, $\int_s^\cdot d\bar{K}_r^n \rightarrow \int_s^\cdot dR_r^+$ in ucp on $[s, \tau_s)$ for every $s \in [0, T)$, where $\tau_s = \inf\{t > s; \Delta R_t^+ > 0\}$, and $Y_t^{n,m} \rightarrow Y_t$, $t \in [0, T]$, as $n, m \rightarrow \infty$.*

Proof. The existence of a solution $(\bar{Y}^n, \bar{M}^n, \bar{A}^n)$ of (3.2) follows from Theorem 4.1. Let

$$\delta_k = \inf\{t \geq 0; \int_0^t |f(r, X_r)| dr \geq k\}.$$

Repeating the proof of Theorem 3.3 one can first prove the existence of solutions of $\text{RBSDE}(\xi, f + dV, L, U)$ on the intervals $[0, \delta_k]$. Then using these solutions and the fact that $\{\delta_k\}$ is stationary one can construct a solution on the whole interval $[0, T]$ (see the proof of Theorem 4.1 for details). \square

Proposition 4.3. *Let (Y, M, R) be a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. Then for any càdlàg processes \hat{L}, \hat{U} such that $L_t \leq \hat{L}_t \leq Y_t \leq \hat{U}_t \leq U_t$ for a.e. $t \in [0, T]$,*

$$\Delta R_{\tau_s}^+ = ({}^pY_{\tau_s} + {}^pV_{\tau_s} - (\bar{L}_{\tau_s-} + V_{\tau_s-}))^- \quad \text{on } \{\tau_s < \infty\},$$

$$\Delta R_{\sigma_s}^- = ({}^pY_{\sigma_s} + {}^pV_{\sigma_s} - (\bar{U}_{\sigma_s-} + V_{\sigma_s-}))^+ \quad \text{on } \{\sigma_s < \infty\}$$

for every $s \in [0, T)$, where

$$\tau_s = \inf\{t > s; \Delta R_t^+ > 0\}, \quad \sigma_s = \inf\{t > s; \Delta R_t^- > 0\}.$$

Proof. Follows directly from the definition of a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. \square

Corollary 4.4. *Let (Y, M, R) be a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. Assume that $L_T \leq \xi \leq U_T$, U, L are càdlàg, \mathcal{F} is quasi-left continuous and the jumps of L, UV are totally inaccessible. Then R is continuous.*

Theorem 4.5. *Let (Y, M, R) be a solution of $\text{RBSDE}(\xi, f + dV, L, U)$ with L of the form*

$$L_t = L_0 - \int_0^t dA_r + \int_0^t dN_r, \quad t \in [0, T] \quad (4.2)$$

for some $A \in \mathcal{V}_0$, $N \in \mathcal{M}_{0,loc}$. Then

$$dR_t^+ \leq \mathbf{1}_{\{Y_{t-}=L_{t-}\}}(f(t, L_t) dt + dV_t^p - dA_t^p)^+,$$

where V^p, A^p are dual predictable projections of V and A , respectively.

Proof. By the Tanaka-Meyer formula (see [22, Theorem IV.70]),

$$\begin{aligned} (Y_t - L_t)^+ &= (Y_0 - L_0)^+ - \int_0^t \mathbf{1}_{\{Y_{r-} > L_{r-}\}} f(r, Y_r) dr - \int_0^t \mathbf{1}_{\{Y_{r-} > L_{r-}\}} d(V_r - A_r - R_r^-) \\ &\quad - \int_0^t \mathbf{1}_{\{Y_{r-} > L_{r-}\}} dR_r^+ - \frac{1}{2} L_t^0(S) + J_t + \int_0^t \mathbf{1}_{\{Y_{r-} > L_{r-}\}} d(M_r - N_r), \end{aligned}$$

where

$$J_t^+ = \sum_{0 \leq s \leq t} (\varphi(S_s) - \varphi(S_{s-}) - \varphi'(S_{s-}) \Delta S_s), \quad S_t = Y_t - L_t, \quad \varphi(x) = x^+$$

and φ' is the left derivative of φ . Observe that J is an increasing process. By the definition of solution of RBSDE, $S_t \geq 0$ for $t \in [0, T]$. Therefore we conclude from the preceding equation and (4.2) that

$$\begin{aligned} &\int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} f(r, Y_r) dr + \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} d(V_r - A_r - R_r^-) \\ &\quad + \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} dR_r^+ + \frac{1}{2} L_t^0(S) + J_t - \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} d(M_r - N_r) = 0. \end{aligned}$$

By the definition of a solution of RBSDE, $\int_0^t dR_r^+ = \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} dR_r^+$. Hence

$$\begin{aligned} \int_0^t dR_r^+ + \frac{1}{2} L_t^0(S) + J_t^p &= - \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} f(r, Y_r) dr \\ &\quad + \int_0^t \mathbf{1}_{\{Y_{r-}=L_{r-}\}} d(R_r^- - V_r^p + A_r^p), \end{aligned}$$

which leads to the desired estimate, because dR^+, dR^- are orthogonal. \square

5 BSDEs with two reflecting barriers: the case of $p \in (1, 2]$

In this section we show some integrability properties of solutions of reflected BSDEs under the assumption that the data are in L^p with $p \in (1, 2]$. Except for Proposition 5.1 and Lemma 5.2 we always assume that the underlying filtration is quasi-left continuous.

Proposition 5.1. *Assume that $M \in \mathcal{M}_{0,loc}$, $K \in \mathcal{V}_0$, X_0 is \mathcal{F}_0 measurable and*

$$X_t = X_0 + \int_0^t dK_r + \int_0^t dM_r, \quad t \in [0, T].$$

Then for $p \in (1, 2)$,

$$\begin{aligned} |X_t|^p - |X_s|^p &\geq p \int_s^t |X_{r-}|^{p-1} \hat{X}_{r-} dK_r + p \int_s^t |X_{r-}|^{p-1} \hat{X}_{r-} dM_r \\ &\quad + \frac{1}{2} p(p-1) \int_s^t \mathbf{1}_{\{X_r \neq 0\}} |X_r|^{p-2} d[X]_r^c \\ &\quad + \sum_{s < r \leq t} (\Delta |X_r|^p - p |X_{r-}|^{p-1} \hat{X}_{r-} \Delta X_r), \end{aligned}$$

where $\hat{x} = \frac{x}{|x|} \mathbf{1}_{\{x \neq 0\}}$, $x \in \mathbb{R}$.

Proof. Write $u_\varepsilon^p(x) = (|x|^2 + \varepsilon^2)^{p/2}$, $x \in \mathbb{R}$. It is easily checked that

$$\frac{du_\varepsilon^p}{dx}(x) = pu_\varepsilon^{p-2}(x)x, \quad \frac{d^2u_\varepsilon^p}{dx^2}(x) = pu_\varepsilon^{p-2}(x) + p(p-2)u_\varepsilon^{p-4}(x)x^2, \quad x \in \mathbb{R}.$$

By the Itô-Meyer formula,

$$\begin{aligned} u_\varepsilon^p(X_t) - u_\varepsilon^p(X_s) &= \int_s^t \frac{du_\varepsilon^p}{dx}(X_{r-}) dX_r + \frac{1}{2} \int_s^t \frac{d^2u_\varepsilon^p}{dx^2}(X_r) d[X]_r^c \\ &\quad + \sum_{s < r \leq t} (\Delta u_\varepsilon^p(X_r) - \frac{du_\varepsilon^p}{dx}(X_{r-}) \Delta X_r) \\ &= \int_s^t pu_\varepsilon^{p-2}(X_{r-}) X_{r-} dK_r + \int_s^t pu_\varepsilon^{p-2}(X_{r-}) X_{r-} dM_r \\ &\quad + \frac{1}{2} \int_s^t (pu_\varepsilon^{p-2}(X_r) + p(p-2)u_\varepsilon^{p-4}(X_r)X_r^2) d[X]_r^c \\ &\quad + \sum_{s < r \leq t} (\Delta u_\varepsilon^p(X_r) - pu_\varepsilon^{p-2}(X_{r-}) X_{r-} \Delta X_r). \end{aligned} \tag{5.1}$$

It is clear that

$$u_\varepsilon^p(X_t) - u_\varepsilon^p(X_s) \rightarrow |X_t|^p - |X_s|^p. \tag{5.2}$$

Observe that $pu_\varepsilon^{p-2}(x)x \rightarrow p|x|^{p-1}\hat{x}$, $x \in \mathbb{R}$, and, by convexity of u_ε^p , $\Delta u_\varepsilon^p(X_r) - pu_\varepsilon^{p-2}(X_{r-})X_{r-}\Delta X_r \geq 0$, $r \in [0, T]$. Therefore applying Fatou's lemma we get

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \sum_{s < r \leq t} (\Delta u_\varepsilon^p(X_r) - pu_\varepsilon^{p-2}(X_{r-})X_{r-}\Delta X_r) \\ \geq \sum_{s < r \leq t} (\Delta |X_r|^p - p|X_{r-}|^{p-1}\hat{X}_{r-}\Delta X_r). \end{aligned} \tag{5.3}$$

By the Lebesgue dominated convergence theorem,

$$\int_s^t pu_\varepsilon^{p-2}(X_{r-})X_{r-} dK_r \rightarrow \int_s^t p|X_{r-}|^{p-1}\hat{X}_{r-} dK_r \tag{5.4}$$

and

$$\int_s^t pu_\varepsilon^{p-2}(X_{r-})X_{r-} dM_r \rightarrow \int_s^t p|X_{r-}|^{p-1}\hat{X}_{r-} dM_r. \tag{5.5}$$

From the identity

$$u_\varepsilon^q(x)|x|^2 = u_\varepsilon^{q+2}(x) - \varepsilon^2 u_\varepsilon^q(x), \quad x, q \in \mathbb{R}$$

it follows that

$$\begin{aligned} & \int_s^t (p u_\varepsilon^{p-2}(X_r) + p(p-2) u_\varepsilon^{p-4}(X_r)) d[X]_r^c \\ &= \int_s^t p(p-1) u_\varepsilon^{p-4}(X_r) |X_r|^2 d[X]_r^c + \int_s^t p \varepsilon^2 u_\varepsilon^{p-4}(X_r) d[X]_r^c. \end{aligned} \quad (5.6)$$

We also have

$$\begin{aligned} \int_s^t u_\varepsilon^{p-4}(X_r) |X_r|^2 d[X]_r^c &= \int_s^t \left(\frac{|X_r|}{u_\varepsilon(X_r)} \right)^{4-p} |X_r|^{p-2} \mathbf{1}_{\{X_r \neq 0\}} d[X]_r^c \\ &\nearrow \int_s^t \mathbf{1}_{\{X_r \neq 0\}} |X_r|^{p-2} d[X]_r^c. \end{aligned} \quad (5.7)$$

From (5.1) and (5.2)–(5.7) we deduce the the desired result. \square

Now we are going to prove some a priori estimates for solutions of reflected BSDEs. For this we need the following lemma.

Lemma 5.2. *Let $p \in (1, 2]$ and let $\varphi(x) = |x|^p$, $x \in \mathbb{R}$. Then for every $x, y \in \mathbb{R}$,*

$$\varphi(x) - \varphi(y) - \varphi'(y)(x - y) \geq \frac{1}{2} \mathbf{1}_{\{|x| \vee |y| \neq 0\}} \varphi''(|x| \vee |y|)(x - y)^2.$$

Proof. By using a mollification of φ one can easily show that for $x \neq y$,

$$\varphi(x) - \varphi(y) - \varphi'(y)(x - y) = \left(\int_0^1 \int_0^1 \alpha \varphi''(y + \alpha \beta(x - y)) d\alpha d\beta \right) (x - y)^2. \quad (5.8)$$

For $\varepsilon > 0$ set $\varphi_\varepsilon^p(z) = (|z|^2 + \varepsilon^2)^{p/2}$. A direct computation shows that

$$\begin{aligned} \frac{d^2 \varphi_\varepsilon^p}{dz^2}(z) &= p(p-1) \varphi_\varepsilon^{p-4}(z) z^2 + \varepsilon^2 \varphi_\varepsilon^{p-4}(z) \\ &\geq p(p-1) \varphi_\varepsilon^{p-4}(z) z^2 = (z^2 + \varepsilon^2)^{(p-2)/2} - \varepsilon^2 (z^2 + \varepsilon^2)^{(p-4)/2}. \end{aligned}$$

Let $z \neq 0$ and $x_1 \leq z \leq x_2$. Then

$$\frac{d^2 \varphi_\varepsilon^p}{dz^2}(z) \geq p(p-1) ((|x_1| \vee |x_2|)^2 + \varepsilon^2)^{(p-2)/2} - \varepsilon^2 (z^2 + \varepsilon^2)^{(p-4)/2}.$$

Since $\frac{d^2 \varphi_\varepsilon^p}{dz^2}(z) \rightarrow \varphi''(z)$ for $z \neq 0$, letting $\varepsilon \rightarrow 0$ in the above inequality we get

$$\varphi''(z) \geq p(p-1) (|x_1| \vee |x_2|)^{p-2} = \varphi''(|x_1| \vee |x_2|).$$

From this we conclude that if $x \neq 0$ or $y \neq 0$ then

$$\int_0^1 \int_0^1 \alpha \varphi''(y + \alpha \beta(x - y)) d\alpha d\beta \geq \frac{1}{2} \varphi''(|x| \vee |y|).$$

This when combined with (5.8) gives the desired result. \square

Let us consider the following hypothesis.

- (A) There exist $\lambda \geq 0$, $\mu \in \mathbb{R}$ and a non-negative progressively measurable process f_t such that for every $y \in \mathbb{R}$,

$$\hat{y}f(t, y) \leq f_t + \mu|y|, \quad dt \otimes dP\text{-a.s.},$$

$$\text{where } \hat{y} = \frac{y}{|y|} \mathbf{1}_{\{y \neq 0\}}.$$

In the remainder of this section we assume that \mathcal{F} is quasi-left continuous.

Proposition 5.3. *Assume (A) and that $\xi \in L^p(\mathcal{F}_T)$, $f_t \in L^p(\mathcal{F})$, $V \in \mathcal{V}_0^p$, $Y \in \mathcal{S}^p$ for some $p \in (1, 2]$. Moreover, assume that Y is a semimartingale and denote by M its martingale part in the Doob-Meyer decomposition. Write $(Y^\alpha, Z^\alpha) = (e^{\alpha t} Y_t, e^{\alpha t} Z_t)$, $dV_t^\alpha = e^{\alpha t} dV_t$ and*

$$f^\alpha(t, y, z) = e^{\alpha t} f(t, e^{-\alpha t} y, e^{-\alpha t} z) - \alpha y.$$

Then if

$$\begin{aligned} & |Y_s^\alpha|^p + \frac{1}{2}p(p-1) \int_s^t |Y_r^\alpha|^{p-2} \mathbf{1}_{\{Y_r^\alpha \neq 0\}} d[Y^\alpha]_r^c + \sum_{s < r \leq t} (\Delta |Y_r^\alpha|^p - p|Y_{r-}^\alpha|^{p-1} \hat{Y}_{r-}^\alpha \Delta Y_r^\alpha) \\ & \leq |Y_t^\alpha|^p + p \int_s^t |Y_r^\alpha|^{p-1} \hat{Y}_r^\alpha f^\alpha(r, Y_r^\alpha, Z_r^\alpha) dr + p \int_s^t |Y_{r-}^\alpha|^{p-1} \hat{Y}_{r-}^\alpha dV_r^\alpha \\ & \quad - p \int_s^t |Y_{r-}^\alpha|^{p-1} \hat{Y}_{r-}^\alpha dM_r, \quad 0 \leq s \leq t \leq T \end{aligned} \tag{5.9}$$

for some $\alpha \geq \mu$ then there is $C > 0$ depending only on p such that

$$\begin{aligned} & E \sup_{t \leq T} |Y_t^\alpha|^p + E \left(\int_0^T d[M]_r \right)^{p/2} \\ & \leq CE \left(|\xi^\alpha|^p + \left(\int_0^T d|V^\alpha|_r \right)^p + \left(\int_0^T f_r^\alpha dr \right)^p \right), \end{aligned}$$

where $\xi^\alpha = e^{\alpha T} \xi$, $f_r^\alpha = e^{\alpha t} f_r$.

Proof. For simplicity we assume that $\alpha = 0$. We only consider the case $p \in (1, 2)$. The case $p = 2$ is included in the assertion of Proposition 6.1. By assumption (A), for every $\tau \in \mathcal{T}$ we have

$$\begin{aligned} & |Y_t|^p + \frac{1}{2}p(p-1) \int_t^\tau |Y_r|^{p-2} \mathbf{1}_{\{Y_r \neq 0\}} d[Y]_r^c + \sum_{t < r \leq \tau} (\Delta |Y_r|^p - p|Y_{r-}|^{p-1} \hat{Y}_{r-} \Delta Y_r) \\ & \leq |Y_\tau|^p + p \int_t^\tau (|Y_r|^{p-1} f_r + \mu|Y_r|^p) dr + p \int_t^\tau |Y_r|^{p-1} d|V|_r \\ & \quad - p \int_t^\tau |Y_r|^{p-1} \hat{Y}_r dM_r. \end{aligned} \tag{5.10}$$

By Lemma 5.2,

$$\begin{aligned} & \sum_{t < r \leq \tau} (\Delta |Y_r|^p - p|Y_{r-}|^{p-1} \hat{Y}_{r-} \Delta Y_r) \\ & \geq \frac{1}{2}p(p-1) \sum_{t < r \leq \tau} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-2} |\Delta Y_r|^2 \\ & = \frac{1}{2}p(p-1) \int_t^\tau \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-2} d[Y]_r^d. \end{aligned}$$

The above inequality when combined with (5.10) and the fact that $\mu \leq \alpha \leq 0$ gives

$$\begin{aligned} & |Y_t|^p + \frac{1}{2}p(p-1) \int_t^\tau \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-2} d[Y]_r \\ & \leq |Y_\tau|^p + p \int_t^\tau |Y_r|^{p-1} f_r dr + \int_t^\tau |Y_r|^{p-1} d|V|_r - p \int_t^\tau |Y_r|^{p-1} \hat{Y}_r dM_r. \end{aligned} \quad (5.11)$$

Since the filtration is quasi-left continuous, $d[Y] \geq d[M]$. Thus

$$\begin{aligned} & E \int_t^\tau \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-1} d[Y]_r \\ & \geq E \int_t^\tau \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-1} d[M]_r. \end{aligned} \quad (5.12)$$

For $k \in \mathbb{N}$ write $\tau_k = \sigma_k \wedge \delta_k$, where $\{\sigma_k\}$ is a fundamental sequence for the local martingale $\int_0^\cdot |Y_r|^{p-1} \hat{Y}_r dM_r$ and

$$\delta_k = \inf\{t \geq 0, \int_0^t (|Y_r| \vee |Y_{r-}|)^{p-2} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} d[M]_r \geq k\}.$$

Substituting (5.12) into (5.11) and then replacing τ by τ_k in (5.11), integrating and letting $k \rightarrow \infty$ we get

$$E|Y_t|^p + \frac{1}{4}p(p-1)E \int_0^T \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} (|Y_r| \vee |Y_{r-}|)^{p-2} d[M]_r \leq EX, \quad (5.13)$$

where $X = |\xi|^p + p \int_0^T |Y_r|^{p-1} f_r dr + p \int_0^T |Y_r|^{p-1} d|V|_r$. Furthermore, by (5.11),

$$\begin{aligned} E \sup_{t \leq T} |Y_t|^p & \leq EX + pE \sup_{t \leq T} \left| \int_t^T |Y_{r-}|^{p-1} \hat{Y}_r dM_r \right| \\ & \leq EX + c_1 p E \sup_{t \leq T} |Y_t|^{p/2} \left(\int_t^T (|Y_r| \vee |Y_{r-}|)^{p-2} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} d[M]_r \right)^{1/2} \\ & \leq EX + \beta E \sup_{t \leq T} |Y_t|^p + \beta^{-1} c_1 p E \int_t^T (|Y_r| \vee |Y_{r-}|)^{p-2} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} d[M]_r. \end{aligned}$$

Taking $\beta > 0$ sufficiently small we get

$$E \sup_{t \leq T} |Y_t|^p \leq c_2 EX. \quad (5.14)$$

Therefore

$$\begin{aligned} E \left(\int_0^T d[M]_r \right)^{p/2} & = E \left(\int_0^T (|Y_r| \vee |Y_{r-}| + \varepsilon)^{2-p} (|Y_r| \vee |Y_{r-}| + \varepsilon)^{p-2} d[M]_r \right)^{p/2} \\ & \leq E \left(\sup_{t \leq T} |Y_t|^{2-p} + \varepsilon^{2-p} \right)^{p/2} \int_0^T (|Y_r| \vee |Y_{r-}| + \varepsilon)^{p-2} d[M]_r^{p/2} \\ & \leq \left(E \left(\sup_{t \leq T} |Y_t|^{(2-p)} + \varepsilon^{2-p} \right)^{(p/2)(2/p)^*} \right)^{1/(2/p)^*} \\ & \quad \times \left(E \int_0^T (|Y_r| \vee |Y_{r-}| + \varepsilon)^{p-2} d[M]_r \right)^{p/2}, \end{aligned} \quad (5.15)$$

where $(2/p)^*$ is the Hölder conjugate to $2/p$. By (5.13) we may pass in (5.15) to the limit as $\varepsilon \rightarrow 0$. We then get

$$E\left(\int_0^T d[M]_r\right)^{p/2} \leq (E \sup_{t \leq T} |Y_t|^p)^{(2-p)/2} \left(E \int_0^T (|Y_r| \vee |Y_{r-}|)^{p-2} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} d[M]_r\right).$$

By Young's inequality,

$$E\left(\int_0^T d[M]_r\right)^{p/2} \leq \frac{2-p}{2} E \sup_{t \leq T} |Y_t|^p + \frac{p}{2} E \int_0^T (|Y_r| \vee |Y_{r-}|)^{p-2} \mathbf{1}_{\{|Y_r| \vee |Y_{r-}| \neq 0\}} d[M]_r.$$

This and (5.13) imply that $Z \in M^p$ and

$$E \sup_{t \leq T} |Y_t|^p + E\left(\sum_{i=1}^{\infty} \int_0^T |Z_r^i|^2 d\langle M^i \rangle_r\right)^{p/2} \leq c_3 EX. \quad (5.16)$$

Finally, observe that

$$EX \leq \beta E \sup_{t \leq T} |Y_t|^p + \beta^{-1} c_4 E\left(|\xi|^2 + \left(\int_0^T f_r dr\right)^p + \left(\int_0^T d|V|_r\right)^p\right).$$

From this and (5.16) the desired estimate follows. \square

For $p \in (1, 2]$ we will use the following modifications to (H4), (H6):

(H4*) $\xi \in L^p(\mathcal{F}_T)$, $V \in \mathcal{V}_0^p$, $f(\cdot, 0) \in L^p(\mathcal{F})$.

(H6*) There exists $X \in \mathcal{V}^p \oplus \mathcal{M}^p$ such that

$$L_t \leq X_t \leq U_t \quad \text{for a.e. } t \in [0, T], \quad \int_0^T |f(t, X_t)| dt \in L^p(\mathcal{F}_T).$$

Proposition 5.4. *Assume that (H1)–(H3) and (H4*) with $p \in (1, 2]$ are satisfied. Then there exists a solution $(Y, M) \in \mathcal{S}^p \otimes \mathcal{M}_0^p$ of BSDE($\xi, f + dV$). Moreover, $\int_0^T |f(t, Y_t)| dt \in L^p(\mathcal{F}_T)$.*

Proof. By [13, Theorem 2.7] there exists a solution (Y, M) of BSDE($\xi, f + dV$) such that Y is of class (D) and $Y \in \mathcal{S}^q$ for $q \in (0, 1)$. Moreover, $Y^n \rightarrow Y$ in \mathcal{S}^q , $q \in (0, 1)$, where $(Y^n, M^n) \in \mathcal{S}^2 \otimes \mathcal{M}^2$ is a solution of BSDE($\xi^n, f^n + dV^n$) and

$$\xi^n = T_n(\xi), \quad f^n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0)), \quad V_t^n = \int_0^t \mathbf{1}_{\{|V|_s \leq n\}} dV_s.$$

By Proposition 5.3,

$$E \sup_{t \leq T} |Y_t^n|^p \leq CE\left(|\xi^n|^p + \left(\int_0^T |f_n(r, 0)| dr\right)^p + \left(\int_0^T d|V^n|_r\right)^p\right).$$

Letting $n \rightarrow \infty$ shows that $Y \in \mathcal{S}^p$ and

$$E \sup_{t \leq T} |Y_t|^p \leq CE\left(|\xi|^p + \left(\int_0^T |f(r, 0)| dr\right)^p + \left(\int_0^T d|V|_r\right)^p\right).$$

By [13, Lemma 2.5], $M \in \mathcal{M}^p$. Hence, by [13, Lemma 2.3],

$$E\left(\int_0^T |f(r, Y_r)| dr\right)^p \leq CE\left(|\xi|^p + \left(\int_0^T |f|(r, 0) dr\right)^p + \left(\int_0^T d|V|_r\right)^p\right),$$

and the proof is complete. \square

Lemma 5.5. *Assume (H1)–(H3), (H4*), (H6*) with $p \in (1, 2]$ are satisfied. Then there exists a solution (Y, Z, R) of $\text{RBSDE}(\xi, f + dV, L, U)$ such that $Y \in \mathcal{S}^p$ and $\int_0^T |f(t, Y_t)| dt \in L^p(\mathcal{F}_T)$.*

Proof. By Theorem 3.3 there exists a solution (Y, Z, R) of $\text{RBSDE}(\xi, f + dV, L, U)$. By Theorem 3.3, (3.4) and (A1) it is enough to prove the integrability properties of Y stated in the lemma in case (Y, Z, R) is a solution of RBSDE with one reflecting barrier. So, let us assume that (Y, Z, R) is the solution of $\text{RBSDE}(\xi, f + dV, L)$. Then the desired properties of Y follow from Theorem 2.12, (2.20) and Proposition 5.4. \square

Theorem 5.6. *Assume that (H1)–(H3), (H4*) with $p \in (1, 2]$ are satisfied. Then there exists a solution $(Y, M, R) \in \mathcal{S}^p \otimes \mathcal{M}^p \otimes \mathcal{V}_0^p$ of $\text{RBSDE}(\xi, f + dV, L, U)$ iff (H6*) is satisfied.*

Proof. Assume that (H1)–(H3), (H4*) are satisfied and $(Y, M, R) \in \mathcal{S}^p \otimes \mathcal{M}_0^p \otimes \mathcal{V}_0^p$ is a solution of $\text{RBSDE}(\xi, f + dV, L, U)$. Then by [13, Lemma 2.5], $\int_0^T |f(r, Y_r)| dr \in L^p(\mathcal{F}_T)$. Therefore (H6*) is satisfied with $X = Y$. Now assume that (H1)–(H3), (H4*), (H6*) are satisfied. First observe that thanks to (3.6) we may assume that (Y, M, R) is a solution of RBSDE with one reflecting barrier, say lower, i.e. we may assume that R is an increasing process. By Lemma 5.5, $Y \in \mathcal{S}^p$ and $E(\int_0^T |f(r, Y_r)| dr)^p < \infty$. From these properties of Y and the fact that R is predictable it follows that there exists a stationary sequence $\{\tau_k\} \subset \mathcal{T}$ such that

$$E([M]_{\tau_k})^{p/2} + E\left(\int_0^{\tau_k} d|R|_r\right)^p < \infty, \quad k \geq 1.$$

By Itô's formula and Young's inequality,

$$\begin{aligned} E([M]_{\tau_k})^{p/2} &\leq c_p E\left(\sup_{t \leq T} |Y_t|^p + \left(\int_0^{\tau_k} |f(r, Y_r)| dr\right)^p \right. \\ &\quad \left. + \left(\int_0^{\tau_k} d|V|_r\right)^p + \left(\int_0^{\tau_k} |Y_{r-}| d|R|_r\right)^{p/2}\right). \end{aligned}$$

Using once again Young's we see that for every $\alpha > 0$,

$$\begin{aligned} E([M]_{\tau_k})^{p/2} &\leq c_p E\left(\sup_{t \leq T} |Y_t|^p + \left(\int_0^{\tau_k} |f(r, Y_r)| dr\right)^p \right. \\ &\quad \left. + \left(\int_0^{\tau_k} d|V|_r\right)^p + \alpha \left(\int_0^{\tau_k} d|R|_r\right)^p\right). \end{aligned} \quad (5.17)$$

On the other hand, since $R_t = Y_0 - Y_t - \int_0^t f(r, Y_r) dr - \int_0^t dV_r + \int_0^t dM_r$ for $t \in [0, T]$, applying the Burkholder-Davis-Gundy inequality we obtain

$$E\left(\int_0^{\tau_k} d|R|_r\right)^p \leq c_p E\left(\sup_{t \leq T} |Y_t|^p + \left(\int_0^{\tau_k} |f(r, Y_r)| dr\right)^p + \left(\int_0^{\tau_k} d|V|_r\right)^p + ([M]_{\tau_k})^{p/2}\right).$$

The above inequality and (5.17) imply that $M \in \mathcal{M}^p$. Hence $R \in \mathcal{V}_0^p$, because we already know that $Y \in \mathcal{S}^p$. \square

6 Reflected BSDEs with generator depending on z

Assume that \mathcal{F} satisfies the usual conditions and the Hilbert $L^2(\mathcal{F}_T)$ is separable. Then (see [3, 22]) there exists a sequence $\{M^i\} \subset \mathcal{M}_0^2$ such that $\{M^i\}$ are orthogonal, i.e. $EM_T^i M_T^j = 0$ for $i \neq j$, and for every $N \in \mathcal{M}^2$,

$$N_t = N_0 + \sum_{i=0}^{\infty} \int_0^t Z_r^i dM_r^i, \quad t \in [0, T] \quad (6.1)$$

for some sequence $\{Z^i\}$ of predictable processes such that

$$E \sum_{i=0}^{\infty} \int_0^T |Z_t^i|^2 d\langle M^i \rangle_t < \infty.$$

For given $A \in \mathcal{V}_0^1$ let us denote by μ_A the measure on $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ defined as

$$\mu_A(B) = E \int_0^T \mathbf{1}_B(t, w) dA_t, \quad B \in \mathcal{B}([0, T]) \otimes \mathcal{F}_T.$$

It is known that the sequence $\{M^i\}$ may be chosen so that $\mu_{\langle M^i \rangle} \gg \mu_{\langle M^j \rangle}$ for $i < j$. In that case the sequence $\{M^i\}$ is unique in the following sense: if $\{\hat{M}^i\} \subset \mathcal{M}_0^2$ is another sequence satisfying the same conditions as $\{M^i\}$ then $\mu_{\langle M^i \rangle}$ is equivalent to $\mu_{\langle \hat{M}^i \rangle}$ for every $i \in \mathbb{N}$. By using the localization procedure one can show that every locally square integrable \mathcal{F} martingale admits representation (6.1) with $\{Z^i\}$ such that

$$P\left(\sum_{i=1}^{\infty} \int_0^T |Z_t^i|^2 d\langle M^i \rangle_t < \infty\right) = 1. \quad (6.2)$$

Set

$$m^i(t, w) = \frac{d\mu_{\langle M^i \rangle}^c}{dt \otimes dP}(t, w), \quad (t, w) \in [0, T] \times \Omega,$$

where $\mu_{\langle M^i \rangle}^c$ is the absolutely continuous part, with respect to $dt \otimes P$, of the measure $\mu_{\langle M^i \rangle}$. By M^0 we denote the space of all processes $Z = (Z^1, Z^2, \dots)$ such that Z^i is predictable for each $i \in \mathbb{N}$ and (6.2) is satisfied. By M^p , $p \geq 1$, we denote the space

$$M^p = \{Z \in M^0; E\left(\sum_{i=1}^{\infty} \int_0^T |Z_t^i|^2 d\langle M^i \rangle_t\right)^{p/2} < \infty\}.$$

We also use the following notation

$$\|z\|_{M_t}^2 = \sum_{i=1}^{\infty} |z^i|^2 m^i(t, w), \quad z \in \mathbb{R}^{\infty}.$$

Let ξ be an \mathcal{F}_T measurable random variable, $V \in \mathcal{V}_0$, L, U be progressively measurable processes and $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^{\infty} \rightarrow \mathbb{R}$ be such that $f(\cdot, \cdot, y, z)$ is progressively measurable for every $(y, z) \in \mathbb{R} \times \mathbb{R}^{\infty}$. We will need the following hypotheses.

(A1) There is $\mu \in \mathbb{R}$ such that for a.e. $t \in [0, T]$ and every $y, y' \in \mathbb{R}$, $z \in \mathbb{R}^{\infty}$,

$$(f(t, y, z) - f(t, y', z))(y - y') \leq \mu |y - y'|^2.$$

(A2) $[0, T] \ni t \mapsto f(t, y, z) \in L^1(0, T)$ for every $y \in \mathbb{R}$, $z \in \mathbb{R}^\infty$,

(A3) $\mathbb{R} \ni y \mapsto f(t, y, z)$ is continuous for a.e. $t \in [0, T]$ and for every $z \in \mathbb{R}^\infty$,

(A4) There is $\lambda \geq 0$ such that for a.e. $t \in [0, T]$ and every $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^\infty$,

$$|f(t, y, z) - f(t, y, z')| \leq \lambda \|z - z'\|_{M_t}.$$

(A5) $\xi \in L^2(\mathcal{F}_T)$, $V \in \mathcal{V}_0^2$, $f(\cdot, 0, 0) \in L^2(\mathcal{F})$,

(A6) There exists $X \in \mathcal{V}^2 \oplus \mathcal{M}^2$ such that $L_t \leq X_t \leq U_t$ for a.e. $t \in [0, T]$ and $f(\cdot, X, 0) \in L^2(\mathcal{F})$.

(A*) There exist $\lambda \geq 0$, $\mu \in \mathbb{R}$ and a non-negative progressively measurable process f_t such that for every $y \in \mathbb{R}$ and $z \in \mathbb{R}^\infty$,

$$\hat{y}f(t, y, z) \leq f_t + \mu|y| + \lambda\|z\|_{M_t}, \quad dt \otimes dP\text{-a.s.}$$

Definition. We say that a triple (Y, Z, R) is a solution of $\text{RBSDE}(\xi, f + dV, L, U)$ if $Z \in M^0$ and the triple (Y, M, R) , where $M_t = \sum_{i=1}^\infty \int_0^t Z_r^i dM_r^i$, $t \in [0, T]$, is a solution of $\text{RBSDE}(\xi, \hat{f} + dV, L, U)$ with

$$\hat{f}(t, y) = f(t, y, Z_t).$$

Proposition 6.1. Assume that (A*) is satisfied and $\xi \in L^2(\mathcal{F}_T)$, $f_t \in L^2(\mathcal{F})$, $V \in \mathcal{V}_0^2$, $(Y, Z) \in \mathcal{S}^2 \otimes M^0$. Moreover, assume that Y is a semimartingale and its martingale part in the Doob-Meyer decomposition is of the form $M = \sum_{i=1}^\infty \int_0^\cdot Z_r^i dM_r^i$. Write $(Y^\alpha, Z^\alpha) = (e^{\alpha t} Y_t, e^{\alpha t} Z_t)$, $dV_t^\alpha = e^{\alpha t} dV_t$ and

$$f^\alpha(t, y, z) = e^{\alpha t} f(t, e^{-\alpha t} y, e^{-\alpha t} z) - \alpha y.$$

Then if

$$\begin{aligned} |Y_s^\alpha|^2 + \int_s^t d[Y^\alpha]_r &\leq |Y_t^\alpha|^2 + 2 \int_s^t Y_r^\alpha f^\alpha(r, Y_r^\alpha, Z_r^\alpha) dr + 2 \int_s^t Y_{r-}^\alpha dV_r^\alpha \\ &\quad - 2 \sum_{i=1}^\infty \int_s^t Y_{r-}^\alpha Z_r^{\alpha, i} dM_r^i, \quad 0 \leq s \leq t \leq T \end{aligned} \quad (6.3)$$

for some $\alpha \geq \mu + \lambda^2$ then $Z \in M^2$ and there is $C > 0$ such that

$$\begin{aligned} E \sup_{t \leq T} |Y_t^\alpha|^2 + E \left(\sum_{i=1}^\infty \int_0^T |Z_r^{\alpha, i}|^2 d\langle M^i \rangle_r \right) \\ \leq CE \left(|\xi^\alpha|^2 + \left(\int_0^T d|V^\alpha|_r \right)^2 + \left(\int_0^T f_r^\alpha dr \right)^2 \right), \end{aligned}$$

where $\xi^\alpha = e^{\alpha T} \xi$, $f_r^\alpha = e^{\alpha t} f_r$.

Proof. For simplicity we assume that $\alpha = 0$. By assumption (A*), for every $\tau \in \mathcal{T}$ we have

$$\begin{aligned} |Y_t|^2 + \int_s^\tau d[Y^\alpha]_r &\leq |Y_\tau|^2 + 2 \int_t^\tau (|Y_r|f_r + \mu|Y_r|^2) dr + 2 \int_t^\tau |Y_r| d|V|_r \\ &\quad + 2\lambda \int_t^\tau |Y_r| \|Z_r\|_{M_r} dr - 2 \sum_{i=1}^\infty \int_t^\tau Y_r - Z_r^i dM_r^i. \end{aligned} \quad (6.4)$$

We have

$$2\lambda|Y_r| \|Z_r\|_{M_r} \leq 2\lambda^2|Y_r|^2 + \frac{1}{2}\|Z_r\|_{M_r}^2.$$

Since $\mu + \lambda^2 \leq \alpha \leq 0$, from the above inequality and (6.4) it follows that

$$\begin{aligned} |Y_t|^2 + \int_t^\tau d[Y]_r &\leq |Y_\tau|^2 + 2 \int_t^\tau |Y_r|f_r dr + 2 \int_t^\tau |Y_r| d|V|_r \\ &\quad + \frac{1}{2} \int_t^\tau \|Z_r\|_{M_r}^2 dr - 2 \sum_{i=1}^\infty \int_t^\tau Y_r - Z_r^i dM_r^i. \end{aligned} \quad (6.5)$$

It is well known that

$$E \int_t^\tau d[Y]_r = E \int_t^\tau d[M]_r = E \int_t^\tau d\langle M \rangle_r \quad (6.6)$$

For $k \in \mathbb{N}$ write $\tau_k = \sigma_k \wedge \delta_k$, where $\{\sigma_k\}$ is a fundamental sequence for the local martingale $\sum_{i=1}^\infty \int_0^\cdot Y_r - Z_r^i dM_r^i$ and

$$\delta_k = \inf\{t \geq 0, \int_0^t \|Z_r\|_{M_r}^2 dr \geq k\}.$$

Substituting (6.6) into (6.5) and then replacing τ by τ_k in (6.5), integrating and letting $k \rightarrow \infty$ we get

$$E|Y_t|^2 + \frac{1}{2}E \sum_{i=1}^\infty \int_0^T |Z_r^i|^2 d\langle M^i \rangle_r \leq EX, \quad (6.7)$$

where $X = |\xi|^2 + 2 \int_0^T |Y_r|f_r dr + 2 \int_0^T |Y_r| d|V|_r$. Furthermore, by (6.5),

$$\begin{aligned} E \sup_{t \leq T} |Y_t|^2 &\leq EX + 2E \sup_{t \leq T} \left| \sum_{i=1}^\infty \int_t^T Y_r - Z_r^i dM_r^i \right| \\ &\leq EX + c_1 2E \sup_{t \leq T} |Y_t| \left(\sum_{i=1}^\infty \int_t^T |Z_r^i|^2 d\langle M^i \rangle_r \right) \\ &\leq EX + \beta E \sup_{t \leq T} |Y_t|^2 + \beta^{-1} c_1 2E \sum_{i=1}^\infty \int_t^T |Z_r^i|^2 d\langle M^i \rangle_r. \end{aligned}$$

Taking $\beta > 0$ sufficiently small and using (6.7) we obtain

$$E \sup_{t \leq T} |Y_t|^p + \frac{1}{2}E \sum_{i=1}^\infty \int_0^T |Z_r^i|^2 d\langle M^i \rangle_r \leq c_2 EX.$$

Combining this with the estimate

$$EX \leq \beta E \sup_{t \leq T} |Y_t|^2 + \beta^{-1} c_4 E \left(|\xi|^2 + \left(\int_0^T f_r dr \right)^2 + \left(\int_0^T d|V|_r \right)^2 \right)$$

we get the desired result. \square

Remark 6.2. If $(Y, Z, R) = (Y^1, Z^1, R^1) - (Y^2, Z^2, R^2)$, where (Y^i, Z^i, R^i) , is a solution of $\text{RBSDE}(\xi^i, f^i + dV^i, L, U)$, $i = 1, 2$, then from Propositions 5.1, 5.3 and condition (c) of the definition of a solution of RBSDE it follows that for every $\alpha \in \mathbb{R}$ the pair (Y^α, Z^α) satisfies (6.3) with $\xi = \xi^1 - \xi^2$, $f(r, y, z) = f^1(r, y + Y_r^2, z + Z_r^2) - f^2(r, Y_r^2, Z_r^2)$, $V = V^1 - V^2$. We will use this fact in the sequel of the paper without further explanations.

Proposition 6.3. Assume (A4). Then there exists at most one solution (Y, Z, R) of $\text{RBSDE}(\xi, f + dV, L, U)$ such that $Y \in \mathcal{S}^2$.

Proof. Follows immediately from Proposition 6.1. \square

Theorem 6.4. Assume (A1)–(A6). Then there exists a solution $(Y, Z, R) \in \mathcal{S}^2 \otimes \mathcal{M}^2 \otimes \mathcal{V}^2$ of $\text{RBSDE}(\xi, f + dV, L, U)$.

Proof. Let us define

$$\Phi = (\Phi^1, \Phi^2) : \mathcal{S}^2 \otimes M^2 \mapsto \mathcal{S}^2 \otimes M^2$$

as follows: for every $(X, H) \in \mathcal{S}^2 \otimes M^2$, $(\Phi^1(X, H), \Phi^2(X, H))$ are the first two components of the solution of $\text{RBSDE}(\xi, f_H + dV, L, U)$ with $f_H(t, x, y) = f(t, x, y, H_t)$. By Theorem 5.6 the mapping Φ is well defined. Let $(X^i, H^i) \in \mathcal{S}^2 \otimes M^2$, $i = 1, 2$, and let $(X, H) = (X^1, H^1) - (X^2, H^2)$, $(Y^i, Z^i) = \Phi(X^i, H^i)$, $i = 1, 2$ and $(Y, Z) = (Y^1, Z^1) - (Y^2, Z^2)$. Observe that

$$Y_t = \int_t^T F(r, Y_r) dr + \int_t^T dR_r^1 - dR_r^2 - \sum_{i=1}^{\infty} \int_t^T Z_r^i dM_r^i, \quad t \in [0, T],$$

where

$$F(r, y) = f(r, y + Y_r^2, H_r^1) - f(r, Y_r^2, H_r^2)$$

and R^i is the finite variation process such that the triple (Y^i, Z^i, R^i) is a solution of $\text{RBSDE}(\xi, f_H + dV, L, U)$, $i = 1, 2$. By Proposition 6.1 and (A4),

$$\begin{aligned} E \sup_{t \leq T} |Y_t|^2 + E \left(\sum_{i=1}^{\infty} \int_0^T |Z_r^i|^2 d\langle M^i \rangle_r \right) &\leq CE \left(\int_0^T |F(r, 0)| dr \right)^2 \\ &\leq \lambda CE \left(\int_0^T \|H_r\|_{M_r} dr \right)^2 \leq \lambda^2 CTE \left(\sum_{i=1}^{\infty} \int_0^T |H_r^i|^2 d\langle M^i \rangle_r \right). \end{aligned} \quad (6.8)$$

It follows that Φ is a contraction on $\mathcal{S}^2 \otimes M^2$ for a sufficiently small T , so using Banach's principle we can construct unique solutions on small intervals. Therefore dividing the interval $[0, T]$ into a finite number of small intervals and using the standard arguments we can construct a solution (Y, Z, R) of $\text{RBSDE}(\xi, f + dV, L, U)$ on the whole interval $[0, T]$. Of course $(Y, Z) \in \mathcal{S}^2 \otimes M^2$. Consequently, $R \in \mathcal{V}_0^2$ by Theorem 5.6. \square

To define solutions of equations with generators depending on z one can use other than (6.1) types of representation theorems for martingales. One possibility is outlined below.

Remark 6.5. Put $E = \mathbb{R}^l \setminus \{0\}$ for some $l \geq 1$. Let B be a d -dimensional Wiener process and N be an independent of B Poisson random measure on $\mathbb{R}_+ \times E$ with the compensator $\nu(dt, de) = dt \otimes \lambda(de)$ such that $\int_E 1 \wedge |e|^2 \lambda(de) < \infty$. For $t \in [0, T]$, $B \in \mathcal{B}(E)$ let us put $\tilde{N}([0, t] \times B) = N([0, T] \times B) - \nu([0, t] \times B)$. It is known (see [26]) that every locally square integrable martingale M has the representation

$$M_t = M_0 + \int_0^t Z_r dB_r + \int_0^t \int_E H_r(e) \tilde{N}(dr, de), \quad t \in [0, T] \quad (6.9)$$

for some predictable \mathbb{R}^d -valued (resp. $L^2(E, \lambda)$ -valued) process Z (resp. H). It is also known that the filtration \mathcal{F} generated by (B, N) is quasi-left continuous. Let

$$f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \lambda) \rightarrow \mathbb{R}$$

be a measurable function such that $f(\cdot, y, z, v)$ is progressively measurable for every $(y, z, v) \in \mathbb{R} \times \mathbb{R}^d \times L^2(E, \lambda)$. After replacing representation (6.1) by (6.9), we may define a solution of $\text{RBSDE}(\xi, f + dV, L, U)$ as a quadruple (Y, Z, H, R) such that the triple (Y, M, R) with M given by (6.9) is a solution of $\text{RBSDE}(\xi, \hat{f} + dV, L, U)$ with

$$\hat{f}(t, y) = f(t, y, Z_t, H_t), \quad (t, y) \in [0, T] \times \mathbb{R}.$$

If we now replace the norm $\|\cdot\|_{M_t}$ by the norm $\|\cdot\|_{\sim}$ on $\mathbb{R}^d \times L^2(E, \lambda)$ given by

$$\|(z, v)\|_{\sim} = |z| + \|v\|_{L^2(E, \lambda)}, \quad (z, v) \in \mathbb{R}^d \times L^2(E, \lambda)$$

and then repeat step by step the proofs of Proposition 6.1 and Theorem 6.4 (with obvious changes) we will get the existence and uniqueness results for solutions of reflected BSDEs in the set-up of the definition given above.

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